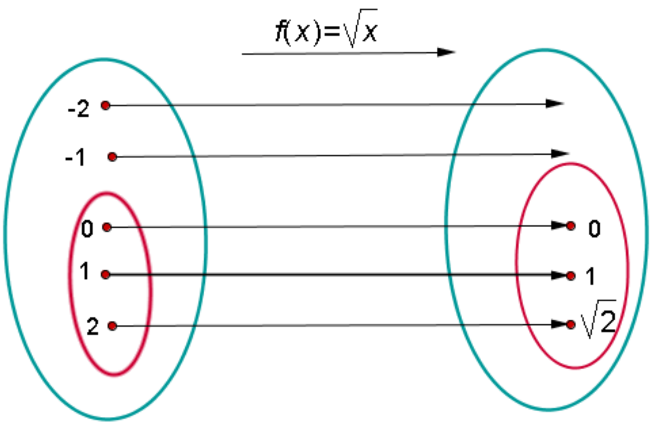
Unit 9. FUNCTIONS.

1. FUNCTION

A real function of real variables is any function, *f*, that associates a real number (image) to each element in a certain subset (domain).

$$f : D \longrightarrow \mathbb{R}$$
$$x \longrightarrow f(x) = y$$

Example:



Initial subset of variables (domain)

Final set of images (range)

The subset which defines the function is called the **domain**.

The number **x** belonging to the domain of the function is called the **independent variable**.

The number **y** associated by the function ***f*** to the independent variable **x** is called the **dependent variable**.

The **image** of x is designated by **f(x)**: *f(x) = y.*

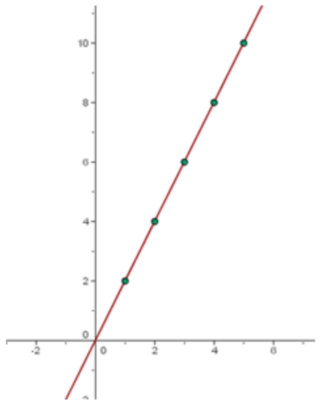
GRAPH OF A FUNCTION:

If *f* is a real function, every pair *(x, y) = (x, f(x))* determined by the function *f* corresponds to the Cartesian plane as a single point P(x, y) = P(x, f(x)). The value of x must belong to the domain of the function.

The set of points belonging to a function is unlimited, and the pairs are arranged in a table of values which correspond to the points of the function. These values, on the Cartesian plane, determine points on the graph. Joining these points with a continuous line gives the graphical representation of the function.

Example 1: f(x) = 2x

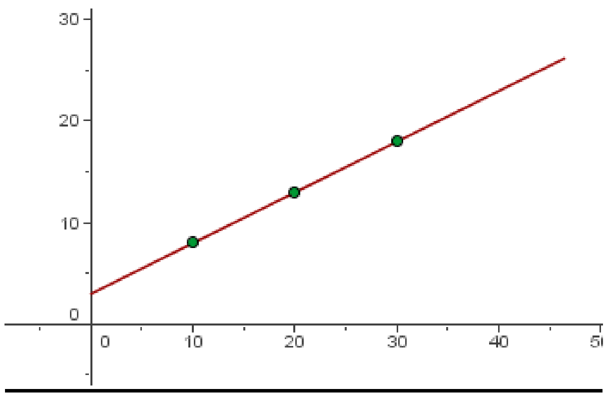
x	1	2	3	4	5
F(x)	2	4	6	8	10



Example 2: f(x) = 3 + ½ x

The price of a taxi ride is represented by: f(x) = 3 + 0.5x, where x is the time in minutes of the ride, and 3 is the minimum fare.

x	10	20	30
y= 3 + 0.5x	8	13	18



2. CHARACTERISTICS OF A FUNCTION

2.1 DOMAIN AND RANGE

The **domain** is the set of elements that have an image:

$$D = \{x \in \mathbb{R} \,/\, \exists\ f(x)\}$$

The **range** is the set of the images:

$$R = \{f(x)/x \in D\}$$

Study the Domain of a Function:

EXAMPLE 1: Domain of a Polynomial Function

The domain is $\mathbb{R}$:

$f(x)= x^2 - 5x + 6$ D = $\mathbb{R}$

EXAMPLE 2: Domain of a Rational Function

The domain is $\mathbb{R}$, minus the values that make the denominator zero (there cannot be a number whose denominator is zero):

$$f(x) = \frac{2x - 5}{x^2 - 5x + 6}$$

$x^2 - 5x + 6 = 0$ D = $\mathbb{R}$ - {2, 3}

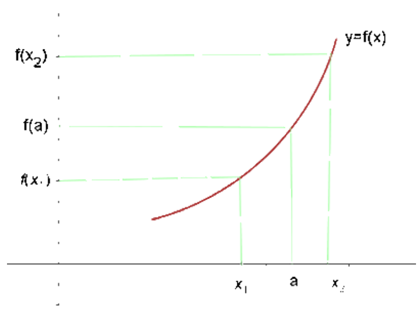
2.2 INCREASING AND DECREASING INTERVALS

A function is **increasing** if when the independent variable (x) increases, the dependent variable (y) also increases.

A function is **decreasing** if when x increases, y decreases.

Many functions have increasing and decreasing intervals.

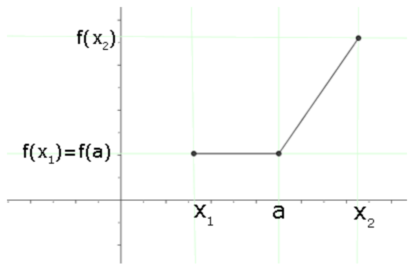
Strictly Increasing Function



$x > a \Rightarrow f(x) > f(a)$

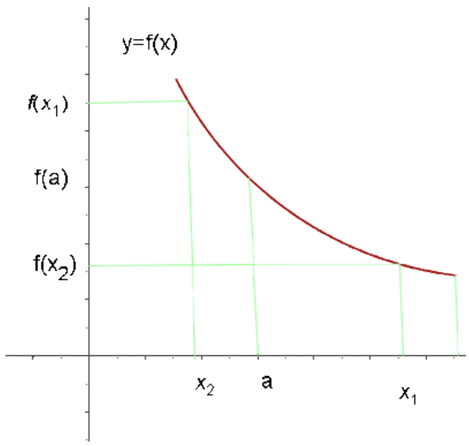
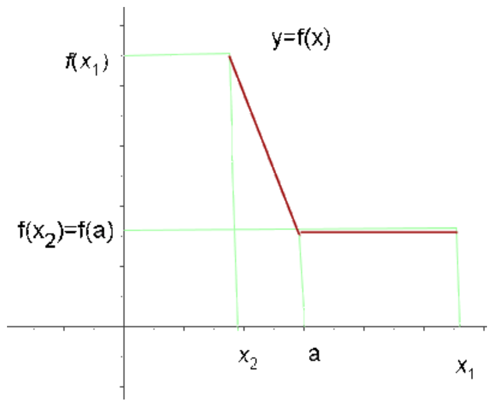
$x < a \Rightarrow f(x) < f(a)$

Increasing Function



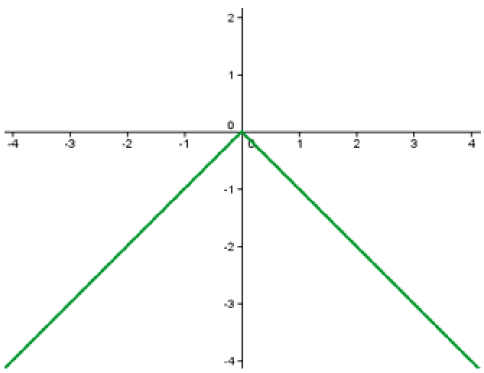
$x > a \Rightarrow f(x) \geq f(a)$

$x < a \Rightarrow f(x) \leq f(a)$

Strictly Decreasing Function	Decreasing Function
 $x > a \Rightarrow f(x) < f(a)$ $x < a \Rightarrow f(x) > f(a)$	 $x > a \Rightarrow f(x) \leq f(a)$ $x < a \Rightarrow f(x) \geq f(a)$

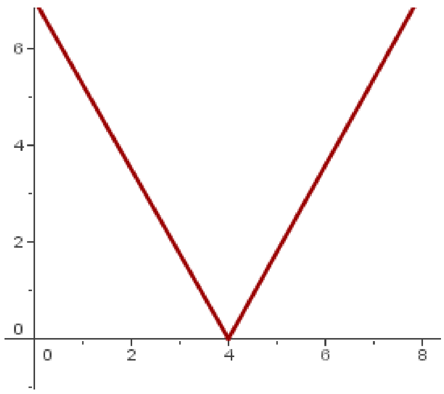
2.3 MAXIMA AND MINIMA

Absolute Maximum: A function has its absolute maximum at x = a if the ordinate in x is greater than or equal to any other point in the domain of the function.



This function has an absolute maximum at (0,0).

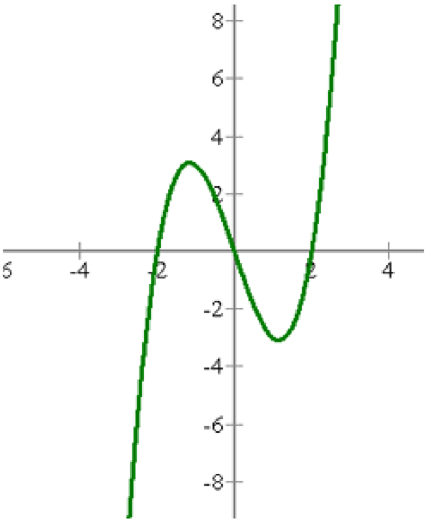
Absolute Minimum: A function has its absolute minimum at x = b if the ordinate at x is less than or equal to any other point in the domain of the function.



This function has an absolute minimum at (4,0).

Relative Maximum: A function, f, has a relative maximum, at point a, if f(a) is greater than or equal to the points near the point a.

Relative Minimum: A function, f, has a relative minimum, at point b, if f(b) is less than or equal to the points near the point b.



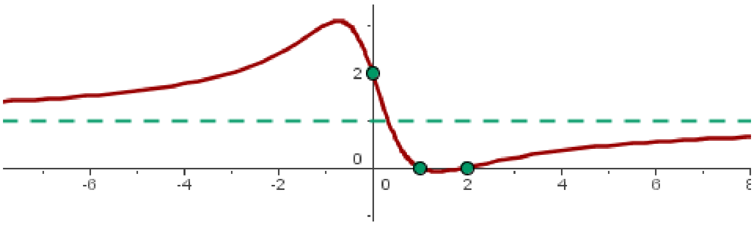
There are a relative maximum at (-1,3) and a relative minimum at (1,-3).

2.4 CONTINUITY AND DISCONTINUITY

A function is said to be **continuous** when you can plot its graph without lifting your pencil off the paper.

When a function contains points at which it is not continuous it is said to be **discontinuous.**

2.5 X- AND Y- INTERCEPT

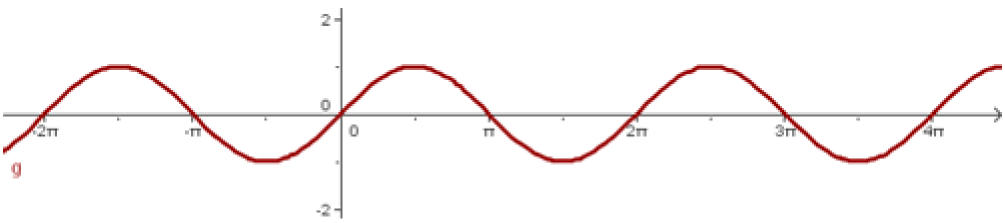


X-intercept points: (1,0); (2,0).

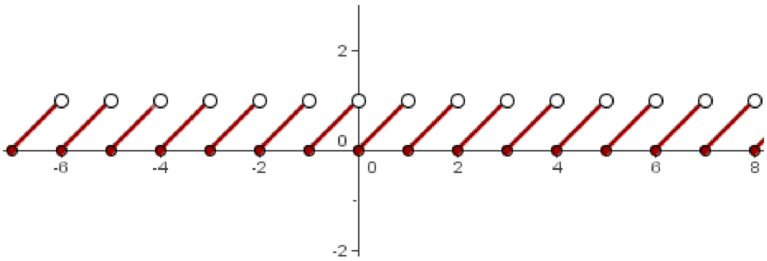
Y-intercept point: (0,2).

2.6 PERIODICITY

A function, f(x), is **periodic** for period T, if it is verified for every integer (z) that f(x) = f(x + zT).

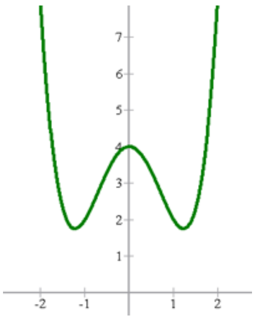
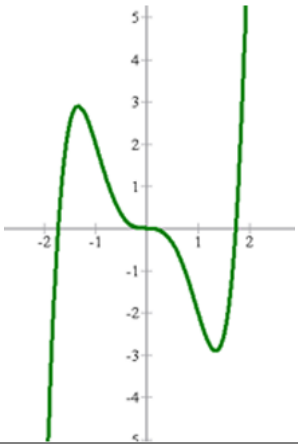


T = 2∏

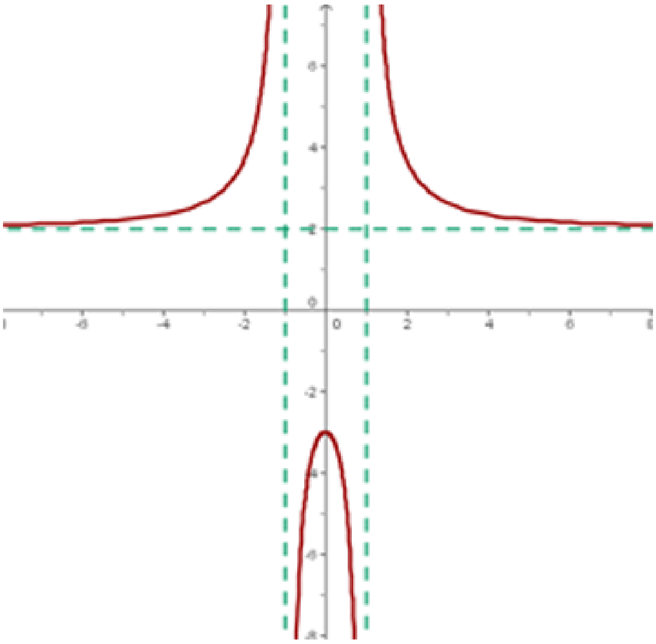


T = 1

2.7 SIMMETRY: EVEN AND ODD FUNCTIONS

Even Function - Symmetry about the Vertical Axis	Odd Function - Symmetry about the Origin
The functions that are symmetrical about the vertical axis are called even functions. 	The functions that are symmetrical about the origin are called odd functions. 

2.8 ASYMPTOTES



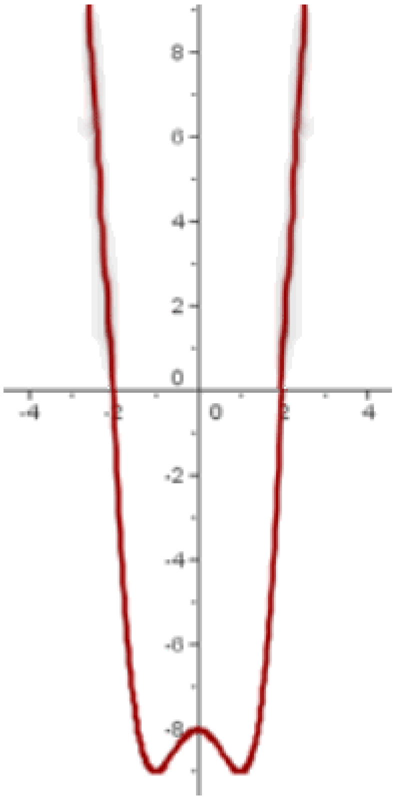
An asymptote (/ˈæsɪmptoʊt/) of a curve is a line such that the distance between the curve and the line approaches zero as they tend to infinity.

Horizontal asymptote: y=2.

Vertical asymptotes: x=-1; x=1.

Example: Study the characteristics of the following functions:

A)



Dom(*f*) = $(-\infty, \infty)$

R(*f*) = $[-9, \infty)$

Increasing intervals: $(-1,0) \cup (1, \infty)$

Decreasing intervals: $(-\infty, -1) \cup (0,1)$

There is no Absolute Maximum

Absolute Minima: (-1,-9) and (1,-9)

Relative Maximum: (0,-8)

Relative Minima: (-1,-9) and (1,-9)

X-intercepts: (-2,0) and (2,0)

Y-intercepts: (0,-8)

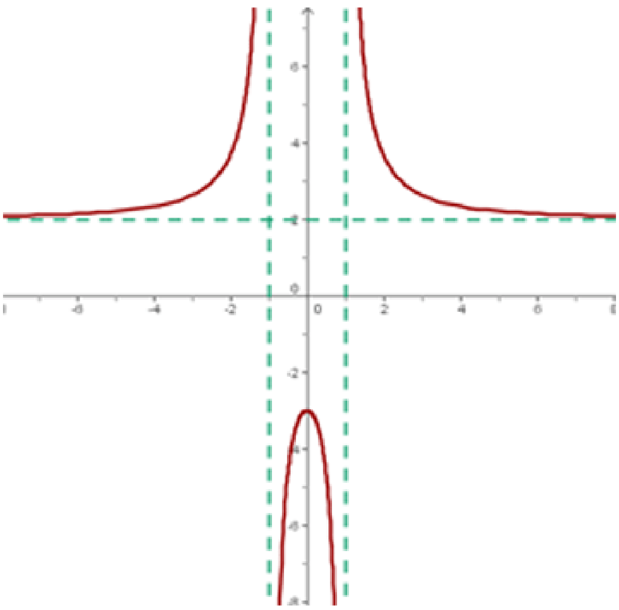
Continuous function

Non-periodic function

Even function

There are no asymptotes

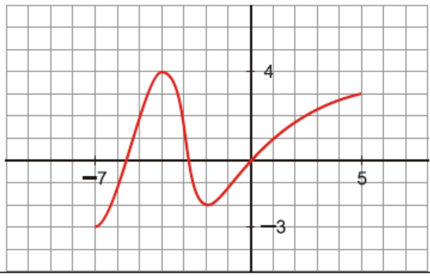
B)



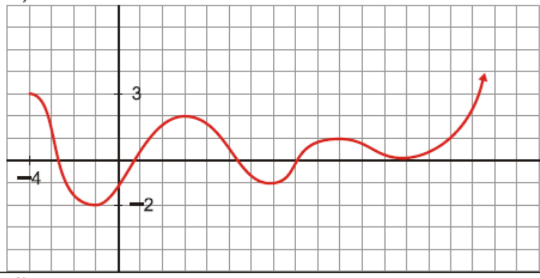
EXERCISE 1:

1.1 Study the characteristics of the following functions:

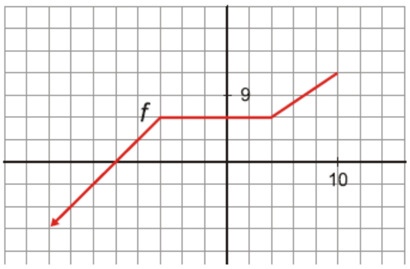
a)



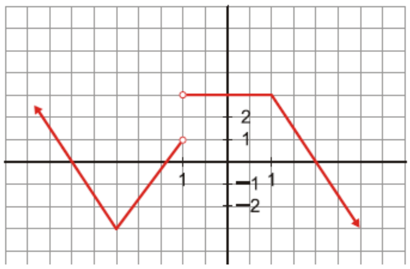
b)



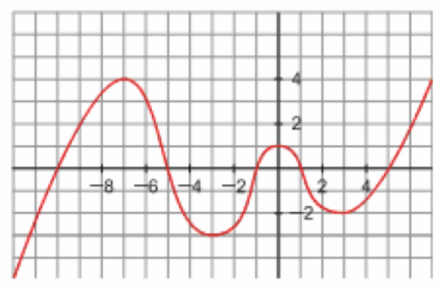
c)



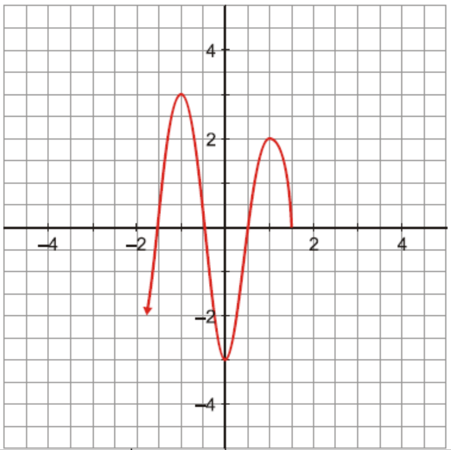
d)



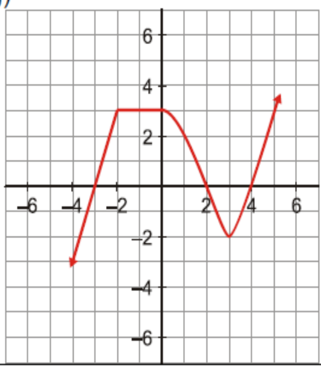
e)



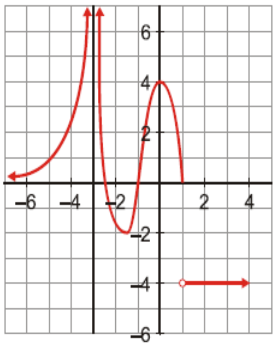
f)



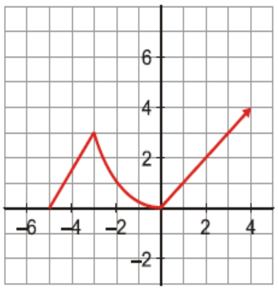
g)



h)



i)



1.2 Calculate the domain of the following functions:

a) $f(x)=\dfrac{1}{x^2-16}$

b) $f(x)=\dfrac{x}{x^2-5x+6}$

c) $f(x)=\dfrac{1}{x^2+4}$

d) $f(x)=\dfrac{x+3}{x^2-2x}$

e) $f(x)=\dfrac{x^2-4}{x+2}$

f) $f(x)=\dfrac{x+1}{x^2}$

3. LINEAR FUNCTIONS

3.1 PROPORTIONAL FUNCTIONS

The equation of a proportional function is y = mx.

The constant, m, can be either positive or negative, and it is called the **slope** of the line. The slope of a line is the amount by which y increases or decreases when x increases by one unit.

If **m>0**, the function is **increasing.** If **m<0**, the function is **decreasing.**

Its graph is a straight line passing through the origin, i.e. the point (0, 0) (see example 1 on page 2).

3.2 FUNCTION y= mx + n

The function y = mx + n is plotted with a **straight line**. Its slope is m. If **m>0**, the function is **increasing.** If **m<0**, the function is **decreasing.**

It intercepts the Y-axis at the point (0, n); n is called the y-intercept (see example 2 on page 2).

When the slope of a line is m = 0 the line takes the form y = n, and runs parallel to the x-axis. It is called a **constant function** because y always has the same value regardless of the value of x.

3.3 LINES FOR WHICH THE SLOPE AND ONE POINT ARE KNOWN

If we know one point of a line (x_0, y_0) as well as its slope, m, we can write its equation as follows:

$$\mathbf{y = y_0 + m(x-x_0)} \leftarrow \text{Point-slope equation}$$

Example: P(1,3) m = 2

y = 3 + 2(x-1) → y = 3 + 2x – 2 → **y = 2x + 1**

3.4 THE EQUATION OF A LINE PASSING THROUGH TWO POINTS

If we know two points on a line we can calculate its slope:

$$\left.\begin{array}{l} P(x_1, y_1) \\ Q(x_2, y_2) \end{array}\right\} \quad m = \frac{change\ in\ y}{change\ in\ x} = \frac{y_2 - y_1}{x_2 - x_1}$$

Using the slope and any one point we can write the point-slope equation of a line.

Example: Find the equation of the line that passes through P(3,5) and Q(-2,4).

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4-5}{-2-3} = \frac{-1}{-5} = \frac{1}{5}$$

$$y = 5 + \frac{1}{5}(x-3) = \frac{1}{5}x + \frac{22}{5}$$

EXERCISE 2:

2.1 Write in each case the equation of the line and graph it:

a) The line has a slope of -3 and its y-intercept is -1.

b) The line has a slope of 4 and it passes through point $(-3, 2)$.

c) The line passes through points $A(-1, 5)$ and $B(3, 7)$.

d) The line passes through point $P(2, -3)$ and it is parallel to the line $y = -x + 1$.

e) The line passes through points $C(3,-1)$ and $D(-3,2)$.

f) The line passes through points $E(5,0)$ y $F(0,2)$.

g) The line passes through point $P(-3, -2)$ and it is parallel to the line $y = 2x - 1$.

2.2 We can buy three pounds of squid at the market for $18. Determine the equation and represent the function that defines the cost of squid depending on weight.

2.3 To hire a car you must pay $100 per day plus $0.30 per mile travelled. Determine the equation of the line that represents the daily cost by the number of miles travelled and graph it. If you travelled a total of 300 miles in one day, how much will you have to pay the rental company?

2.4 It has been observed that a particular plant's growth is directly proportional to time. It measured 2 cm when it arrived at the nursery, and 2.5 cm exactly one week later. If the plant continues to grow at this rate, determine the function that represents the plant's growth; then graph it.

2.5 Calculate “t” in order that these points are all lined up:

1) $A(-3,6)$, $B(0,4)$ and $C(1, t)$

2) $A(-4,-6)$, $B(-1,1)$ and $C(3, t)$

3) $A(1,3)$, $B(2,-2)$ and $C(-5, t)$

4. QUADRATIC FUNCTION

The equation of a quadratic function is: $y = ax^2 + bx + c$

Its graph is a parabola.

Graphical Representation of the Parabola

A parabola can be built from these points:

1) Vertex:

$$x_v = \frac{-b}{2a} \qquad y_v = f\left(\frac{-b}{2a}\right) \qquad V\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

The **axis of symmetry** passes through the **vertex** of the parabola.

The equation of the **axis of symmetry** is: $x = \dfrac{-b}{2a}$

2) x-intercepts:

For the intercept with the x-axis, the **second coordinate** is always **zero**:

$$ax^2 + bx + c = 0$$

To find the x-intecepts, solve the resulting quadratic equation:

Two intercept points: $(x_1, 0)$ $(x_2, 0)$ if $b^2 - 4ac > 0$

One intercept point: $(x_1, 0)$ if $b^2 - 4ac = 0$

No intercept points if $b^2 - 4ac < 0$

3) y-intercept:

For the intercept with the y-axis, the **first coordinate** is always **zero**:

$f(0) = a\cdot 0^2 + b\cdot 0 + c = c \qquad$ **(0, c)**

Example: Graph the quadratic function $y = x^2 - 4x + 3$.

1) Vertex:

$x_v = -(-4)/2 = 2 \qquad y_v = 2^2 - 4\cdot 2 + 3 = -1$

V(2, -1)

2) x-intercepts: $\quad x^2 - 4x + 3 = 0$

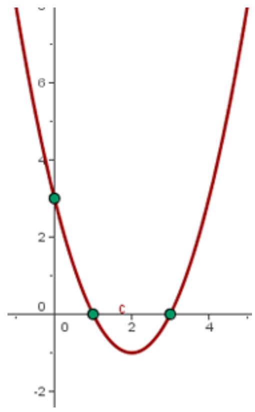
$$x = \frac{4 \pm \sqrt{16-12}}{2} = \frac{4 \pm 2}{2} \qquad \begin{matrix} x_1 = 3 \\ x_2 = 1 \end{matrix}$$

(3, 0) (1, 0)

3) y-intercept:

(0, 3)

SOLUTION:



EXERCISE 3:

3.1 Graph the following functions:

a) $y = -x^2 + 4x - 3$

b) $y = x^2 + 2x + 1$

c) $y = -x^2 + 6x - 5$

d) $y = x^2 - 4x - 5$

3.2 Graph the following quadratic functions, indicating the vertex:

a) $y = x^2$

b) $y = x^2 + 1$

c) $y = x^2 - 2$

d) $y = (x - 2)^2$

e) Could you conclude something from the previous graphs?

3.3 From the graph of function $f(x) = x^2$, graph the following translations:

a) $y = x^2 - 3$

b) $y = (x + 1)^2$

c) $y = (x + 1)^2 - 3$

3.4 A small plane flies between Cádiz and Ceuta. Its flight height is determined by the rule: $h(x) = -30x^2 + 900x$, where $h(x)$ is the height of the plane in metres at x minutes after take-off from Cádiz. Draw the graph and work out:

a) The maximum height the plane reaches.

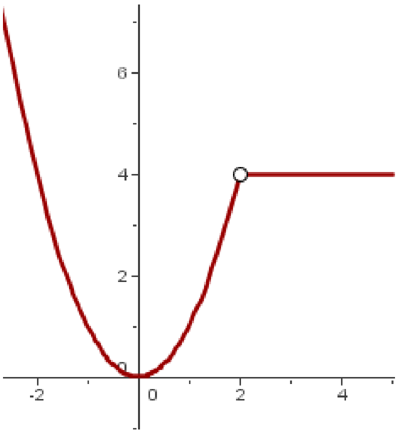
b) The duration of the flight

5. PIECEWISE FUNCTIONS

Piecewise functions are functions defined by different criteria according to the intervals being considered.

Example:

$$y = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x > 2 \end{cases}$$



EXERCISE 5:

5.1 Draw the graph of the following functions:

a) $$f(x) = \begin{cases} 2x+4 & \text{if} & x > 0 \\ 4-2x & \text{if} & x < 0 \end{cases}$$

b) $$f(x) = \begin{cases} 1 & \text{if} & x \le 1 \\ x & \text{if} & 1 < x \le 3 \\ -x+6 & \text{if} & 3 < x \le 6 \\ 0 & \text{if} & 6 < x \end{cases}$$

c) $$f(x) = \begin{cases} -3 & si\ x \le 0 \\ x-3 & si\ 0 \le x \le 5 \\ 2 & si\ x \ge 5 \end{cases}$$

d) $$f(x) = \begin{cases} -1-x & si\ x < -1 \\ 1-x^2 & si\ -1 \le x \le 1 \\ x-1 & si\ x > 1 \end{cases}$$

e) $$f(x) = \begin{cases} 4-x^2 & si\ x \le 1 \\ x+2 & si\ x > 1 \end{cases}$$

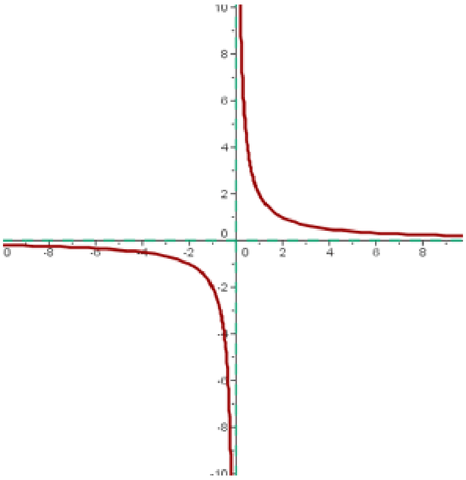
6. RATIONAL FUNCTIONS

6.1. INVERSELY RATIONAL FUNCTION

The **inversely proportional function** is $y = \frac{k}{x}$. Its graph is a hyperbola.

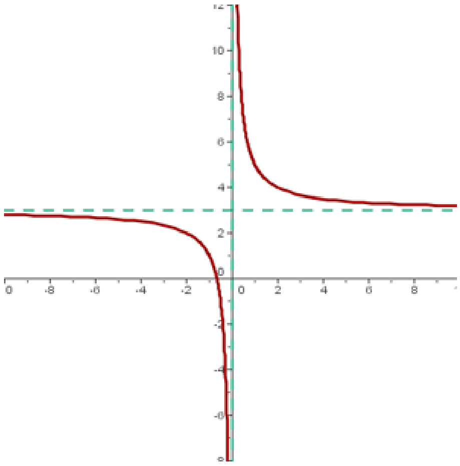
This function has two asymptotes, the axes. The centre of the hyperbola, which is where the asymptotes intersect, is the origin. This function is an odd function.

Example: $y = \frac{2}{x}$



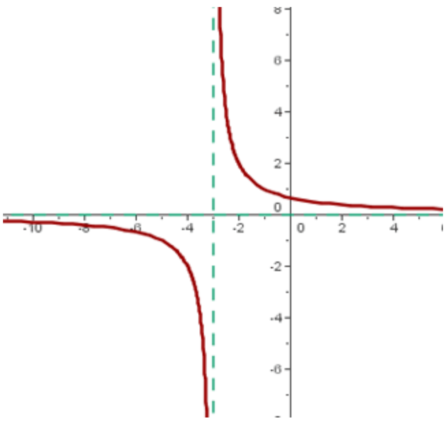
EXAMPLE: VERTICAL TRANSLATION

$$y = \frac{2}{x} + 3$$



EXAMPLE: HORIZONTAL TRANSLATION

$$y = \frac{2}{x+3}$$

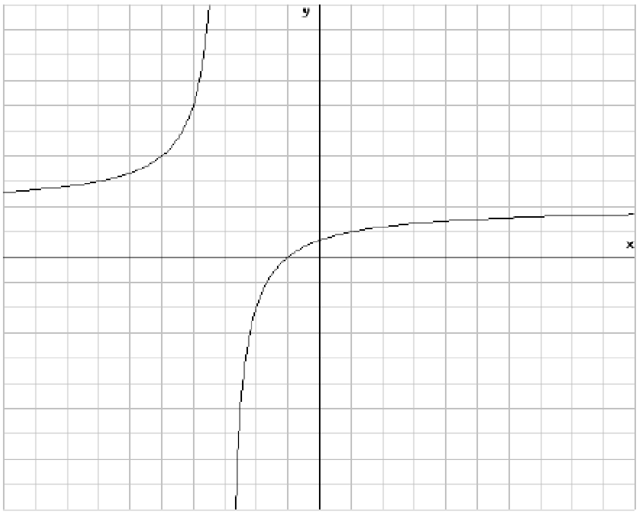


6.2. GENERAL EXPRESSION OF A HYPERBOLA

$$y = \frac{ax + b}{cx + d}$$

Example:

$$f(x) = \frac{2x+2}{x+3}$$



EXERCISE 6:

6.1 Graph the following rational functions:

a) $y = \frac{6}{x}$

b) $y = \frac{6}{x} - 3$

c) $y = \frac{6}{x-3}$

d) $y = \frac{6}{x+2} - 1$

e) $f(x) = \frac{-1}{x}$

f) $f(x) = \frac{-1}{x+1}$

g) $f(x) = \frac{-1}{x-2} + 1$

6.2 Draw the graph of the inversely proportional function $y = \frac{-4}{x}$

a) Domain and Range
b) Asymptotes
c) Intervals of increasing or decreasing
d) Continuity

e) Sketch the function $y = \frac{-4}{x+2}$. Are there any asymptotes?

6.3 Graph the following rational functions:

a) $f(x) = \dfrac{3x-3}{x+2}$

b) $f(x) = \dfrac{x}{x-3}$

c) $f(x) = \dfrac{x+3}{x-1}$

7. IRRATIONAL FUNCTIONS

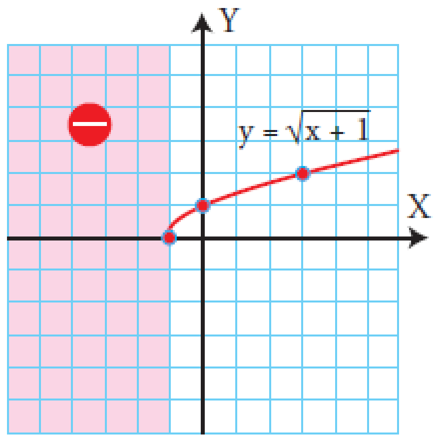
$f(x) = \sqrt{x+1}$

First, calculate the domain of the function:

$x + 1 \geq 0 \Rightarrow x \geq -1 \Rightarrow \mathrm{Dom}(f) = [-1, +\infty)$

Then, obtain several points in the plane:

x	-1	0	3	…
$y = \sqrt{x+1}$	0	1	2	…



EXERCISE 7:

7.1 Graph the following irrational functions:

a) $f(x) = \sqrt{x-2}$

b) $f(x) = \sqrt{x+4}$

c) $f(x) = -\sqrt{x+3}$

d) $f(x) = \sqrt{x}$

e) $f(x) = \sqrt{x} + 2$

8. EXPONENTIAL AND LOGARITHMIC FUNCTIONS

8.1. EXPONENTIAL FUNCTION

$f(x) = a^x, \quad a > 0, a \neq 1$

The characteristics of an exponential function are:

a) $Dom(f) = (-\infty, \infty)$

b) $R(f) = (0, \infty)$

c) Continuous function

d) The function passes through point (1,a)

e) The X-axis is a horizontal asymptote

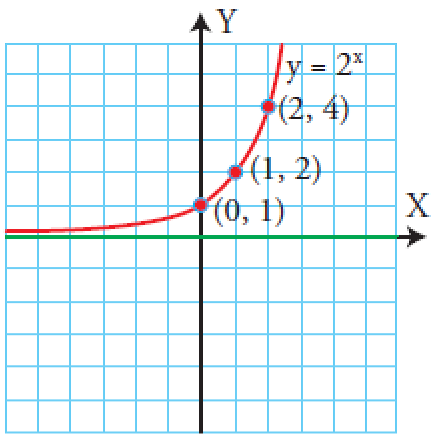
f) Y-intercept: (0,1)

g) If a>1, then the function is increasing. If 0<a<1, then the function is decreasing.

Examples:

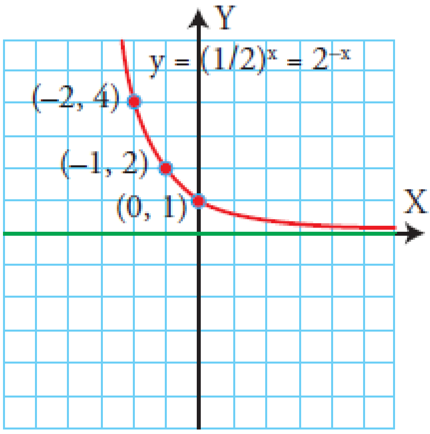
A) $f(x) = 2^x$

x	…	−3	−2	−1	0	1	2	3	…
$y = 2^x$	…	1/8	1/4	1/2	1	2	4	8	…



B) $f(x) = \left(\dfrac{1}{2}\right)^x = 2^{-x}$

x	…	−3	−2	−1	0	1	2	3	…
$y = (1/2)^x$	…	8	4	2	1	1/2	1/4	1/8	…



8.2. LOGARITHMIC FUNCTION

$f(x) = \log_a x, \quad a > 0, a \neq 1$

The characteristics of a logarithmic function are:

a) $Dom(f) = (0, \infty)$

b) $R(f) = (-\infty, \infty)$

c) Continuous function

d) The function passes through point (a,1)

e) The Y-axis is a vertical asymptote

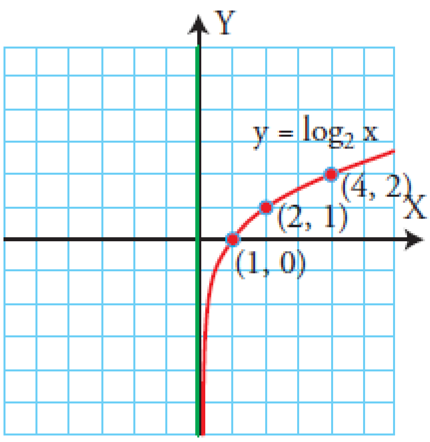
f) X-intercept: (1,0)

g) If a>1, then the function is increasing. If 0<a<1, then the function is decreasing.

Examples:

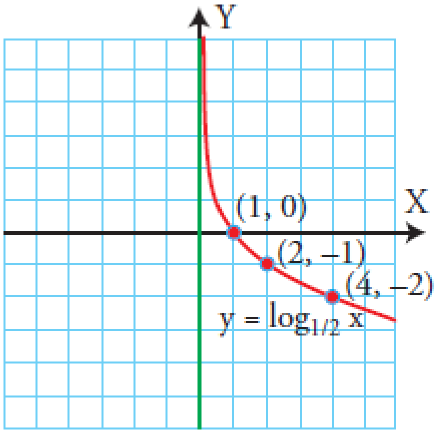
A) $f(x) = \log_2 x$

x	…	1/8	1/4	1/2	1	2	4	8	…
$\mathbf{y = \log_2 x}$	…	−3	−2	−1	0	1	2	3	…



B) $f(x) = \log_{\frac{1}{2}} x$

x	...	1/8	1/4	1/2	1	2	4	8	...
$y = \log_{1/2} x$	...	3	2	1	0	–1	–2	–3	...



EXERCISE 8:

8.1 Graph these functions:

a) $f(x) = 2^x + 3$

b) $f(x) = 2^{x-4}$

c) $f(x) = 3^x - 1$

d) $f(x) = \left(\frac{1}{3}\right)^x + 2$

8.2 Graph these functions:

a) $f(x) = \log_2 x + 1$

b) $f(x) = \log_2 (x+1)$

c) $f(x) = \log_{\frac{1}{2}} (x-1)$

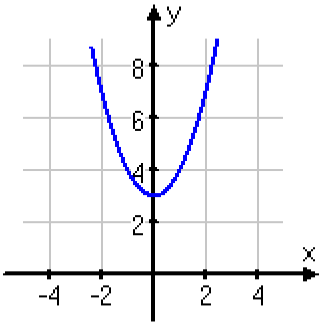
d) $f(x) = \log_3 x$

9. FUNCTION TRANSFORMATIONS

A function transformation (or translation) takes a basic function f(x) and then "transforms" it or "translates" it, which means that you change the formula a bit and then the graph moves around.

9.1. VERTICAL TRANSLATION

For instance, the graph for $x^2 + 3$ looks like this:



This is three units higher than the basic quadratic, $f(x) = x^2$. That is, $x^2 + 3$ is $f(x) + 3$. We added a "3" outside the basic squaring function $f(x) = x^2$ and so we obtained the transformed function $x^2 + 3$ from the basic quadratic x^2.

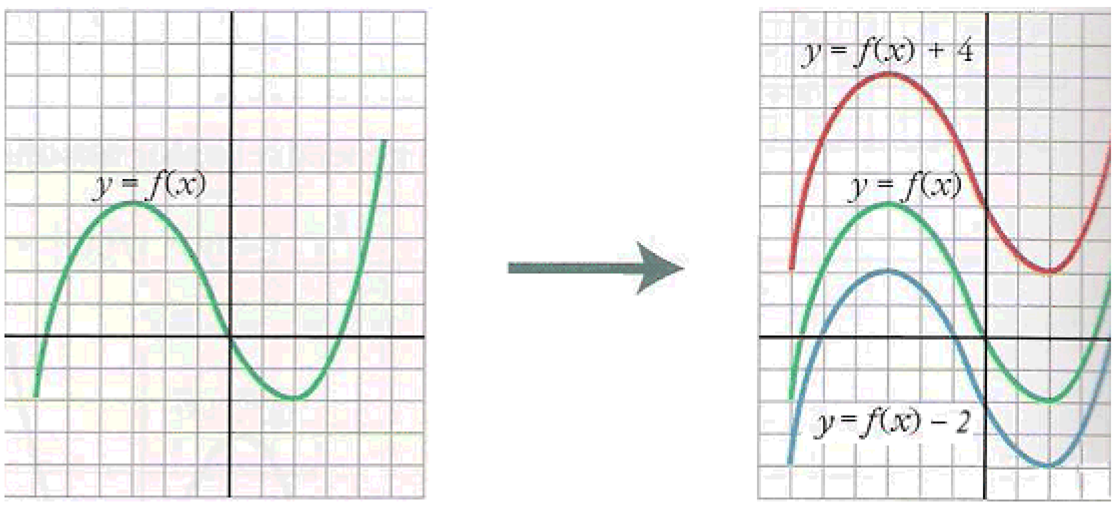
This is always true: To move a function up, you add outside the function: $f(x) + b$ is $f(x)$ moved up b units. Moving the function down works the same way; $f(x) – b$ is $f(x)$ moved down b units.

Summarising:

• $f(x) + a$ is $f(x)$ shifted upward by **a** units.

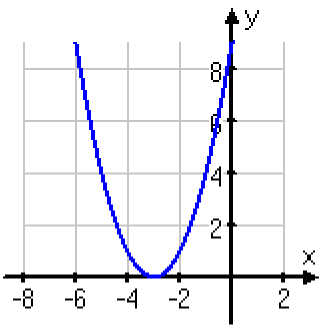
• $f(x) – a$ is $f(x)$ shifted downward by **a** units.

Example:



HORIZONTAL TRANSLATION

On the other hand, $(x + 3)^2$ looks like this:



In this graph, f(x) has been moved over three units to the left: $f(x + 3) = (x + 3)^2$ is f(x) shifted three units to the left.

This is always true: To shift a function left, add inside the function's argument: f(x + b) gives f(x) shifted b units to the left. Shifting to the right works the same way, but you have to subtract from f(x), so f(x – b) is f(x) shifted b units to the right.

Warning: Don't be tempted to think that f(x + 3) moves f(x) to the right by three, because "+3" is to the right; it is the other way round: adding moves the graph to the left; subtracting moves it to the right.

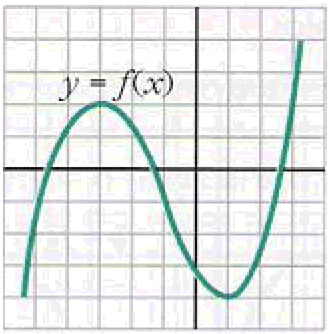
If you find it confusing, think about the point on the graph where x = 0. For f(x + 3), if you want y to be 0, does x need to be to the left or to the right of 0? In this case, x needs to be –3, so the argument is –3 **+ 3** = 0, so I need to add 3 in order to move the function left by three. This process will tell you where the x-values, and thus the graph, have shifted.

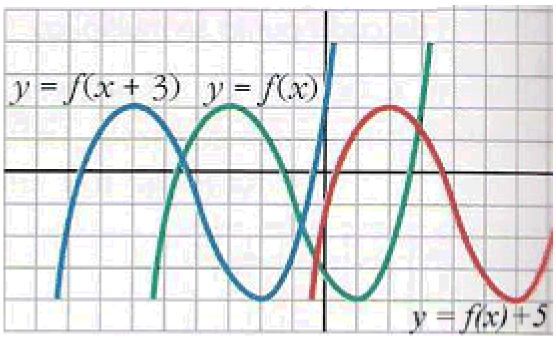
Summarising:

• f(x + a) is f(x) shifted left by **a** units.

• f(x – a) is f(x) shifted right by **a** units.

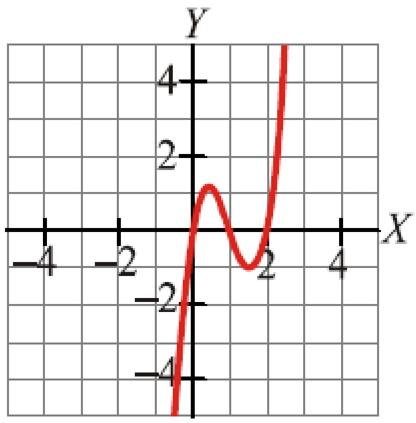
Example:





EXERCISE 9:

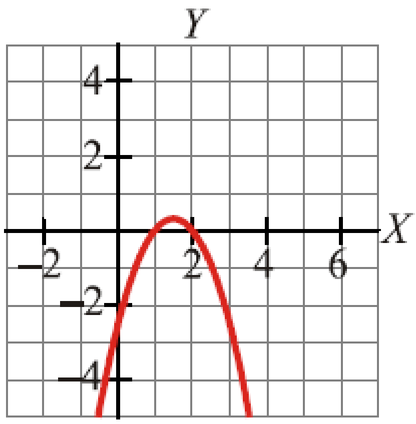
9.1 This is the graph of y = f(x):



a) Graph y = f(x) – 3

b) Graph y = f(x + 2)

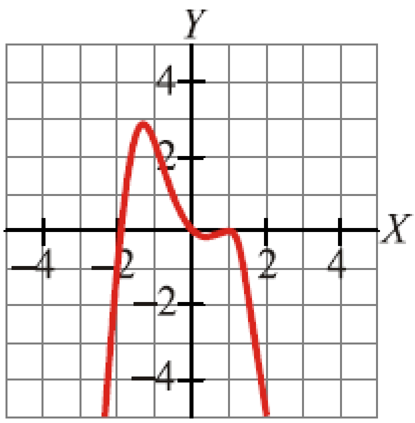
9.2 This is the graph of y = f(x):



a) Graph y = f(x) + 2

b) Graph y = f(x - 3)

9.3 This is the graph of y = f(x):



a) Graph y = f(x) + 1

b) Graph y = f(x + 1)