Unit 6. INEQUALITY AND SYSTEMS OF INEQUALITIES

1. LINEAR INEQUALITY

An inequality is the result of replacing the = sign in an equation with <, >, ≤, or ≥. For example, $3x - 2 < 7$ is a linear inequality. We call it "linear" because if the < were replaced with an = sign, it would be a linear equation.

1.1. LINEAR INEQUALITY IN ONE VARIABLE

A linear inequality in one variable is an expression like this: ***ax + b < 0***, where a and b are real numbers and the operator < could also be >, ≤ or ≥.

The solution of a linear inequality in one variable is **a ray**, that is, a half bounded interval *(=intervalo semiabierto)*: $(a, +\infty)$; $[a, +\infty)$; $(-\infty, a)$; or $(-\infty, a]$, depending on the inequality symbol (>, ≥, <, or ≤, respectively).

A linear inequality in one variable is solved as a first order equation, but it is necessary to flip the inequality symbol when multiplying or dividing by a negative number.

Example: Solve the following inequality

$$\frac{x}{2} + \frac{7}{4} \leq x + 1$$

$$2x + 7 \leq 4(x + 1)$$

$$2x + 7 \leq 4x + 4$$

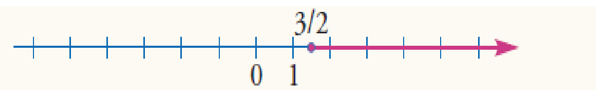
$$2x - 4x \leq 4 - 7$$

$$-2x \leq -3$$

$$2x \geq 3$$

$$x \geq 3/2$$

The solution is the interval:



$$[3/2, +\infty) = \{x \in \mathbb{R}, x \geq 3/2\}$$

1.2. LINEAR SYSTEM OF INEQUALITIES IN ONE VARIABLE

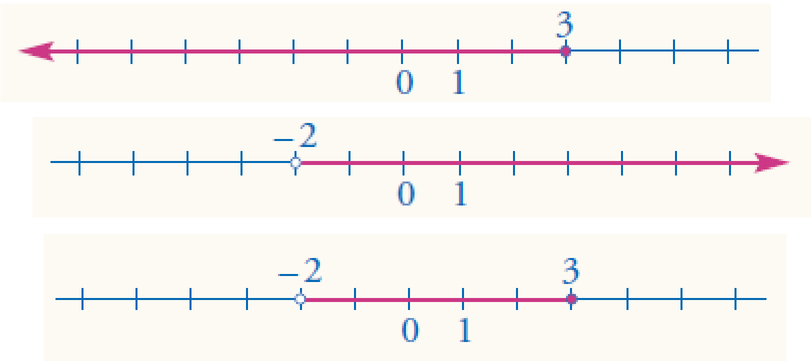
A linear system of inequalities in one variable is solved by solving each inequality separately. The solution of the system is the common part of both solutions.

Example 1: Solve this linear system of inequalities.

$$\left.\begin{array}{l} x-3\leq 0 \\ x+2>0 \end{array}\right\}$$

$x-3\leq 0$
$x\leq 3$
$(-\infty, 3] = \{x\in\mathbb{R}, x\leq 3\}$

$x+2>0$
$x>-2$
$(-2, +\infty) = \{x\in\mathbb{R}, x>-2\}$



The solution of the system is the interval:

$(-2, 3] = \{x\in\mathbb{R}, -2<x\leq 3\}$

Example 2: Solve this linear system of inequalities:

$$\left.\begin{array}{l} x+3\geq 0 \\ 2x-5\leq 0 \end{array}\right\}$$

1.3. ABSOLUTE VALUE INEQUALITY

An inequality with an absolute value is an expression like this: $|ax+b|<r$, where the inequality symbol could also be >, ≤ or ≥.

Here are the steps to solve an inequality with an absolute value:

1) Solve the associated equation using an appropriate method. This solution or solutions will make up the set of critical values. At these values, sign changes occur in the inequality.
2) Plot the critical values on a number line. Use closed circles for ≤ and ≥ inequalities, and use open circles o for < and > inequalities.
3) Test each interval defined by the critical values. If an interval satisfies the inequality, then it is part of the solution. If it does not satisfy the inequality, then it is not part of the solution.

Example 1: Solve |2x + 5| > 1 and graph the solution.

1. Solve the related equation |2x + 5| = 1. To solve absolute value equations, you must solve two cases:

$$2x + 5 = 1 \text{ and } -(2x + 5) = 1$$

   A) Solve 2x + 5 = 1

      2x = -4

      x = -4/2 = -2

   B) Solve –(2x + 5) = 1

      -2x – 5 = 1

      -2x = 6

      x = 6/(-2) = -3

2. Plot the critical values, x = -2 and x =-3. Use open circles since this is >.

3. Test arbitrary values of each interval. You may test the value in the original inequality or in its simplified form.

In **Interval 1**, let x=0 in |2x + 5| > 1.

|2·0 + 5| > 1 is TRUE, so Interval 1 is part of the solution.

In **Interval 2**, let x = -2.5 in |2x + 5| > 1.

$|2\cdot(-2.5) + 5| > 1$ is FALSE, so Interval 2 is NOT part of the solution.

In **Interval 3**, let $x = -4$ in $|2x + 5| > 1$.

$|2\cdot(-4) + 5| > 1$ is TRUE, so Interval 3 is part of the solution.

We shade Interval 1 and Interval 3, but do not include the endpoints.

The solution is $x > -2$ or $x < -3$. In interval notation, we would write this as $(-\infty, -3) \cup (-2, \infty)$.

Example 2: Solve $|x - 3| \leq 1$ and graph the solution

2. NONLINEAR INEQUALITY IN ONE VARIABLE

In short, solving inequalities is about one thing: sign changes. We must find all the points at which there are sign changes —we call these points critical values. Then we must determine which, if any, of the intervals bounded by these critical values result in a solution. The solution to the inequality will consist of the set of all points contained by the solution intervals.

2.1. POLYNOMIAL INEQUALITY

A polynomial inequality is an expression as $P(x) < 0$, where $P(x)$ is a polynomial and the inequality symbol could also be $>$, $\leq$ or $\geq$.

The solution of a polynomial inequality could be the empty set, $\emptyset$, a point, an interval, or the union of several intervals.

Method to solve Polynomial Inequalities:

1. Move all terms to one side of the inequality sign. You should have only zero on one side of the inequality sign.

2. Solve the associated equation using an appropriate method. This solution or solutions will make up the set of critical values. At these values, sign changes occur in the inequality.

3. Plot the critical values on a number line. Use closed circles for $\leq$ and $\geq$ inequalities, and use open circles for $<$ and $>$ inequalities.

4. Test each interval defined by the critical values. If an interval satisfies the inequality, then it is part of the solution. If it does not, then it is not part of the solution.

Example 1: Solve $x^2 - 2x - 3 \geq 0$

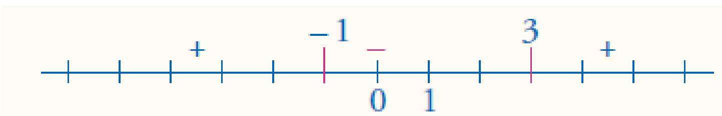
$$x^2 - 2x - 3 = 0$$

$$(x + 1)(x - 3) \geq 0$$

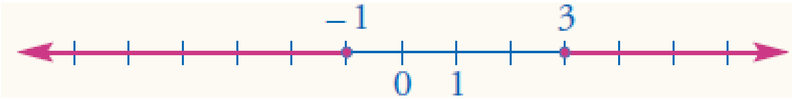
$x=-2 \Rightarrow (-1)\cdot(-5)=5 > 0$ (+)

$x=0 \Rightarrow (1)\cdot(-3)=-3 < 0$ (-)

$x=4 \Rightarrow (5)\cdot(1)=5 > 0$ (+)



The solution is:

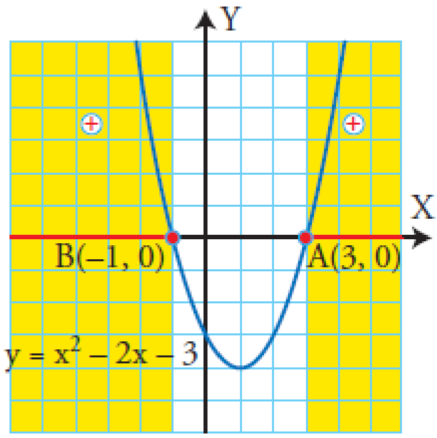


$(-\infty, -1] \cup [3, +\infty)$

Graphical interpretation:

$$x^2 - 2x - 3 \geq 0 \Rightarrow f(x) = x^2 - 2x - 3$$

The previous function represents a parabola. The inequality represents its positive part.



Example 2: $x^2 + 2x < 3$

2.2. RATIONAL INEQUALITY

A rational inequality is an expression such as $\frac{P(x)}{Q(x)} < 0$, where P(x) and Q(x) are polynomials and the inequality symbol could also be >, ≤ or ≥.

Method to solve Rational Inequalities:

1. Move all terms to one side of the inequality sign. You should have only zero on one side of the inequality sign.

2. Find the *critical values* by solving the associated equations to the numerator and denominator. At these values, sign changes occur in the inequality.

3. Plot the critical values on a number line. Always use open circles for < and > inequalities. Use closed circles for ≤ and ≥, unless the value is a root of the denominator —you must always use open circles for values resulting in roots of the denominator, since this value cannot be part of the solution!

4. Test each interval defined by the critical values. If an interval satisfies the inequality, then it is part of the solution. If it does not satisfy the inequality, then it is not part of the solution.

In short, the only difference between solving a polynomial inequality and a rational inequality is that there are additional critical values that are roots of the denominator, and you must never include these additional values as part of the solution, even if it is a ≤ or ≥ inequality. If you include these values you are dividing by zero!

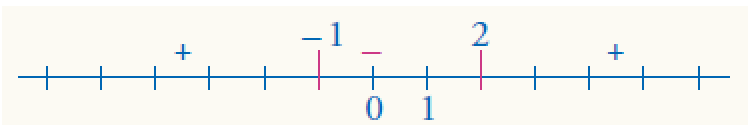
Example 1: $\frac{x+1}{x-2} \geq 0$

x+1=0=>x=-1

x-2=0=>x=2

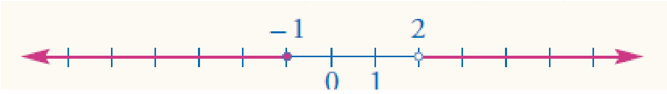
x=-2=> $\frac{-1}{-4} > 0\,(+)$

x=0=> $\frac{1}{-2} < 0\,(-)$



x=3=> $\frac{4}{1} = 4 > 0\,(+)$

The solution is:

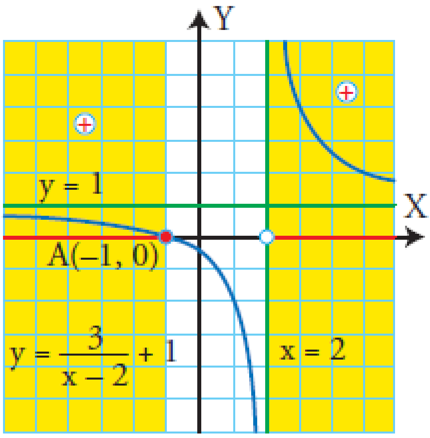


$(-\infty, -1] \cup (2, +\infty)$

Graphical interpretation:

$$\frac{x+1}{x-2} \geq 0 \Rightarrow f(x) = \frac{x+1}{x-2}$$

This function is a hyperbola.



Example 2:

Solve $\frac{x+1}{x-3} - 1 \leq 2$

3. LINEAR INEQUALITY IN TWO VARABLES

A linear inequality in two variables can be written $Ax + By > C$, where the $>$ symbol can also be $\geq$, $<$, or $\leq$, where A, B, and C are real numbers, and where both A and B are different from 0.

The solution of a linear inequality in two variables is a **half-plane** where the straight line $Ax + By = C$ is the boundary. The half-plane contains the straight line if the inequality symbol is $\leq$ or $\geq$.

A linear inequality in two variables is solved by graphing the linear inequality following these steps:

1. Draw the graph of the straight line that is the boundary. Make the line solid if the inequality involves $\leq$ or $\geq$. Make the line dashed if the inequality involves $<$ and $>$.
2. Choose a test point. Choose any point not on the line, and substitute the coordinates of that point in the inequality.
3. Shade the appropriate region. Shade the region that includes the test point if it satisfies the original inequality. Otherwise, shade the region on the other side of the boundary line.

When drawing the boundary line in Step 1, be careful to draw a solid line if the inequality includes equality ($\leq$, $\geq$) or a dashed line if equality is not included ($<$, $>$).

Example 1: Graph $3x + 4y < 12$

1. Solve the inequality for y and graph the boundary line (as a dashed line, because the inequality symbol is $<$).

   $4y < -3x + 12$

   $y < -\frac{3}{4}x + 3$

   Now graph $y = -\frac{3}{4}x + 3$

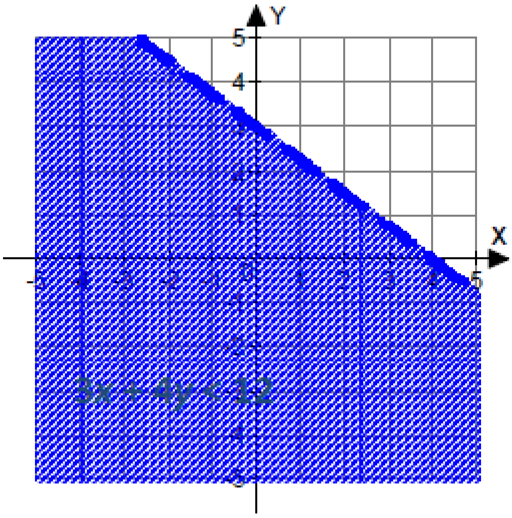
2. Choose a test point not on the line, say (0, 0), and substitute it for x and y in the original inequality.

   $3x + 4y < 12$

   $3(0) + 4(0) < 12$

   $0 < 12$ (True)

3. Shade the half-plane which includes the test point (below the boundary line):



4. LINEAR SYSTEM OF INEQUALITIES IN TWO VARIABLES

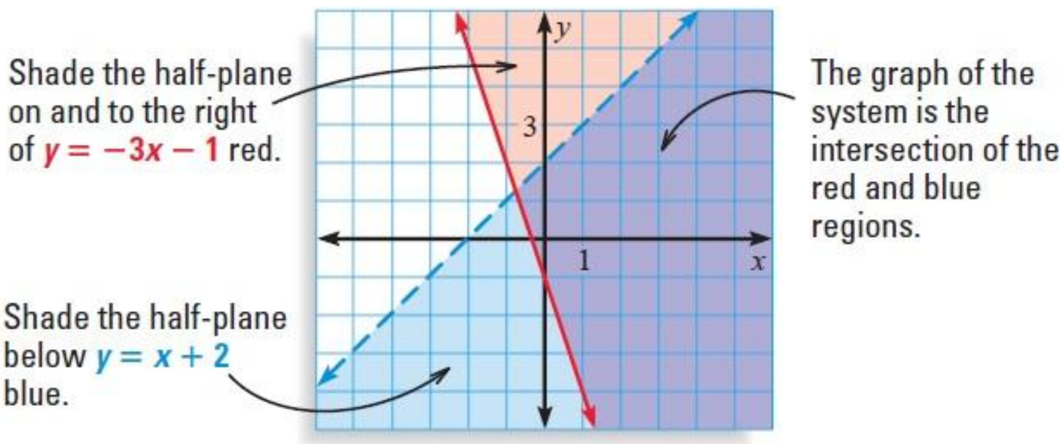
Many practical problems in business, science, and engineering involve systems of linear inequalities. A solution of a system of inequalities in x and y is a point (x, y) that satisfies each inequality in the system.

To graph a system of linear inequalities, do the following for each inequality in the system:

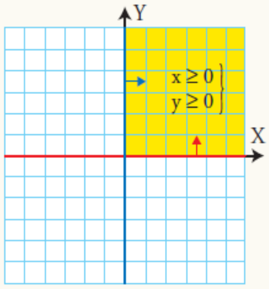
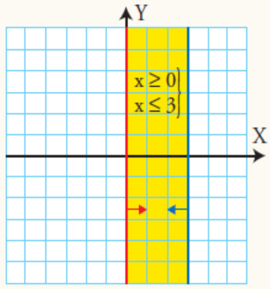
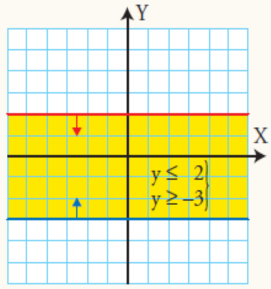
- Graph the lines that correspond to the inequality. Use a dashed line for an inequality with < or >, and a solid line for an inequality with ≤ or ≥.
- Lightly shade the half-planes that are the graph of the inequality. Coloured pencils may help you distinguish the different half-planes.

The graph of the system is the region common to all of the half-planes. If you used coloured pencils, it is the region that has been shaded with every colour. This region represents the **solution set** of the system.

Example 1: Graph the system $\begin{cases} y \geq -3x - 1 \\ y < x + 2 \end{cases}$

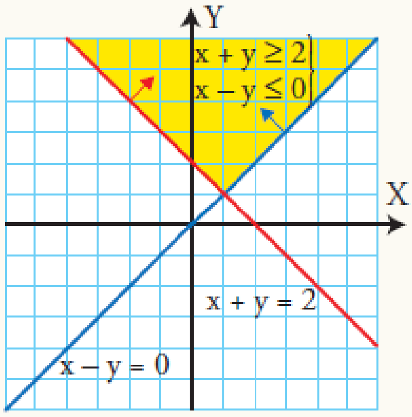


Example 2:

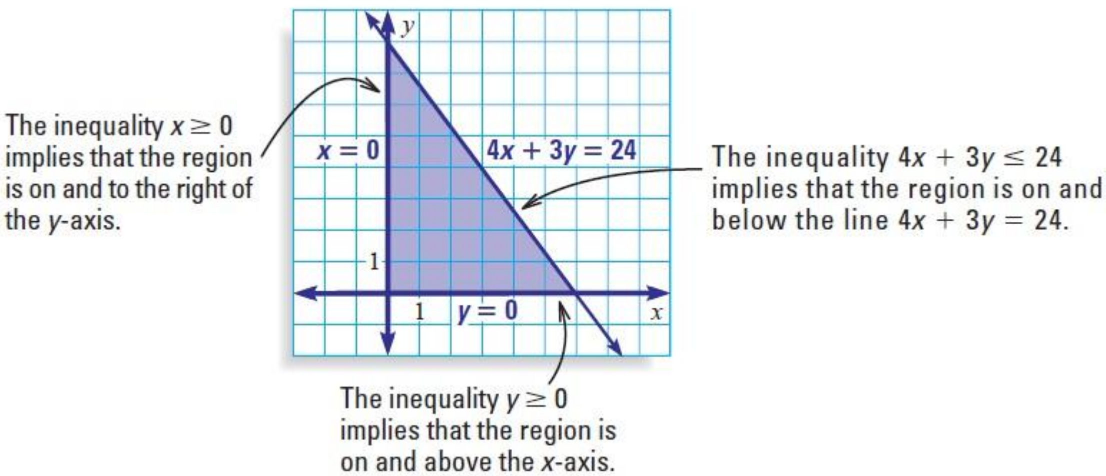
$\begin{rcases} x \geq 0 \\ y \geq 0 \end{rcases}$	$\begin{rcases} x \geq 0 \\ x \leq 3 \end{rcases}$	$\begin{rcases} y \leq 2 \\ y \geq -3 \end{rcases}$
		

Example 3:

$$\begin{rcases} x + y \geq 2 \\ x - y \leq 0 \end{rcases}$$



Example 4: Graph the system $\begin{cases} x \geq 0 \\ y \geq 0 \\ 4x + 3y \leq 24 \end{cases}$



You can use a system of linear inequalities to describe a real-life situation, as shown in the following example.

Example 6:

A person's theoretical maximum heart rate is 220 - x where x is the person's age in years ($20 \le x \le 65$). When a person is exercising, it is recommended that the person tries to reach a heart rate that is at least 70% of the maximum, and at most 85% of the maximum.

a) You are making a poster about health. Write and graph a system of linear inequalities that describes the information given above.

b) A 40-year-old person has a heart rate of 150 (heartbeats per minute) when exercising. Is the person's heart rate in the target zone?

SOLUTION:

Let y represent the person's heart rate. From the given information, you can write the following four inequalities:

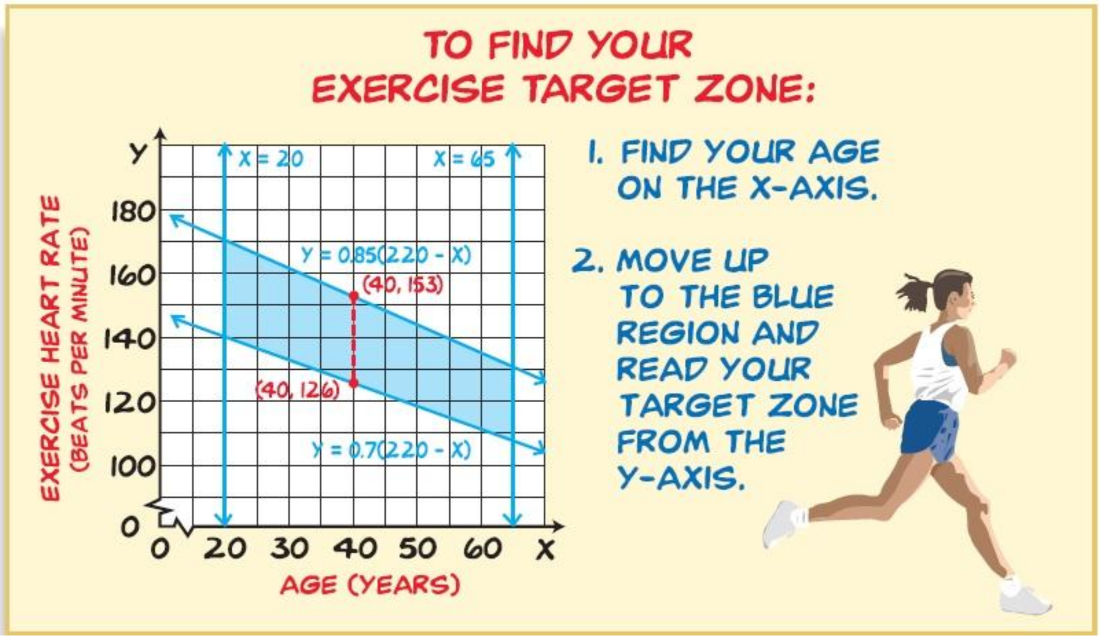
$x \ge 20$ Person's age must be at least 20.

$x \le 65$ Person's age can be at most 65.

$y \ge 0.7(220 - x)$ Target rate is at least 70% of maximum rate.

$y \le 0.85(220 - x)$ Target rate is at most 85% of maximum rate.

The graph of the system is shown below.



From the graph you can see that the target zone for a 40-year-old person is between 126 and 153, inclusive. A 40-year-old person who has a heart rate of 150 is within the target zone.

EXERCISES:

EXERCISE 1 Solve the following inequalities:

a) $5x - 4 < 3x - 1$

b) $2x - 3(x + 2) \leq 2(x - 1) - 1$

c) $x - 2(x - 1) > 10 - 2(x + 3)$

d) $\dfrac{2x}{3} + \dfrac{x + 2}{6} < \dfrac{3x}{2} + 1$

e) $\dfrac{4x + 1}{3} - \dfrac{2x + 1}{2} \leq \dfrac{x}{12} + \dfrac{5}{6}$

f) $\dfrac{x - 1}{2} \leq \dfrac{3x + 10}{5} + \dfrac{5x + 3}{15}$

EXERCISE 2 Solve the following systems of inequalities:

a) $\left.\begin{array}{r} x + 4 > 0 \\ 2x - 3 \leq 1 \end{array}\right\}$

b) $\left.\begin{array}{l} x - 1 \geq 0 \\ x + 2 < 0 \end{array}\right\}$

c) $\left.\begin{array}{l} -13x + 21 \leq 2 - 3(5x - 7) \\ x + 2(3x - 5) > 6x - 7 \end{array}\right\}$

EXERCISE 3 Solve the following inequalities:

a) $|x - 1| < 4$

b) $|x + 3| \leq 2$

c) $|3x - 6| < 3$

d) $|2x - 4| \geq 6$

EXERCISE 4 Solve the following polynomial inequalities:

a) $-x^2 + 6x - 5 \geq 0$

b) $x^2 + x \geq \dfrac{15}{4}$

c) $x^3 - 12x^2 + 41x - 30 < 0$

d) $x^2 - 6x + 8 < 0$

e) $3x^3 + 5x^2 > 4x^2$

f) $3x^2 + 6x + 1 < (x+3)^2$

EXERCISE 5 Solve the following rational inequalities:

a) $\dfrac{x-2}{x-3} \geq 0$

b) $\dfrac{x-4}{x} < 0$

c) $\dfrac{(x+3)(x-3)}{x^2-9} > 0$

d) $\dfrac{5x}{x^2 - x - 6} < 0$

e) $\dfrac{2x^2 - 7x - 15}{x} \leq 0$

EXERCISE 6 Graph these linear inequalities in two variables:

a) $2x + y \le 4$

b) $y < 2$

c) $x \ge 3$

d) $x + y \ge 2$

EXERCISE 7 Solve the following linear systems of inequalities in two variables:

a) $\left.\begin{array}{l} x - y \le 3 \\ x + y \ge 5 \end{array}\right\}$

b) $\left.\begin{array}{l} 2x + 3y > 6 \\ 2x - \ \ y < 6 \end{array}\right\}$

c) $\left.\begin{array}{l} x \ge 0 \\ y \ge 0 \\ x + y \ge 2 \\ x + y \le 5 \end{array}\right\}$

e) $\left.\begin{array}{l} y \ge 0 \\ 3x + 2y \ge 6 \\ -3x + 4y \le 12 \end{array}\right\}$

f)

EXERCISE 8: A shop owner wants to buy refrigerators and washing machines; they cost € 500 and € 400, respectively. If she has only enough space to store 50 appliances, and only € 22,000 to invest, represent on a plane the target zone of all the possible solutions for the number of refrigerators and washing machines that she can buy.

EXERCISE 9. A manufacturer sells chairs and tables. To make them he needs 2 and 5 hours, respectively, of manual labour, plus 1 and 2 hours, respectively, to paint them. If the manufacturer cannot pay the weekly wages for more than 200 hours of manual labour and 90 hours of painting, represent on a plane the target zone of the possible solutions.