



Unit 5. SYSTEMS OF EQUATIONS

1. LINEAR SYSTEMS OF EQUATIONS

Two equations with two unknowns form a system if they have a common solution:

$$\begin{cases} a_1x + b_1y = c_1 \\ a_2x + b_2y = c_2 \end{cases}$$

The solution of a system is a pair of numbers x_1, y_1 , such that when we replace x with x_1 and y with y_1 , both equations are verified.

$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases}$$

The solution to this system is: $x = 2, y = 3$; we can verify it:

$$\begin{cases} 3 \cdot 2 - 4 \cdot 3 = -6 \\ 2 \cdot 2 + 4 \cdot 3 = 16 \end{cases} \quad \begin{cases} 6 - 12 = -6 \\ 4 + 12 = 16 \end{cases} \quad \begin{matrix} -6 = -6 \\ 16 = 16 \end{matrix}$$

Properties of Linear Systems

1. If both members of an equation in a system are added or subtracted the same expression, the resulting system is equivalent.

$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} 3x - 4y + 3 = -6 + 3 \\ x + 2y - 5y = 8 - 5y \end{cases} \quad x = 2, y = 3$$

2. If both members of the equations in a system are multiplied or divided by a nonzero number, the resulting system is equivalent.

$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} 3 \cdot (3x - 4y) = -6 \cdot 3 \\ \frac{2x + 4y}{2} = \frac{16}{2} \end{cases} \quad x = 2, y = 3$$

3. If an equation of a system is added or subtracted another equation of the same system, the resulting system is equivalent.

$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} 3x - 4y = -6 \\ 2x + 4y + 3x - 4y = 16 - 6 \end{cases} \quad x = 2, y = 3$$



4. If an equation in a system is replaced by another equation that is obtained from adding the two equations from a system previously multiplied or divided by a nonzero number, the resulting system is equivalent.

$$\begin{array}{l} \begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} 3x - 4y = -6 \\ \frac{2x + 4y}{2} = \frac{16}{2} \end{cases} \quad \begin{cases} 3x - 4y = -6 \\ x + 2y = 8 \end{cases} \\ \\ \begin{cases} 3x - 4y + x + 2y = -6 + 8 \\ x + 2y = 8 \end{cases} \quad \begin{cases} 4x - 2y = 2 \\ x + 2y = 8 \end{cases} \quad x = 2, \quad y = 3 \end{array}$$

5. If the order of the equations or the order of the unknowns changes, the resulting system is equivalent.

$$\begin{array}{l} \begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} 2x + 4y = 16 \\ 3x - 4y = -6 \end{cases} \quad x = 2, \quad y = 3 \\ \\ \begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases} \quad \begin{cases} -4y + 3x = -6 \\ 4y + 2x = 16 \end{cases} \quad x = 2, \quad y = 3 \end{array}$$

2. LINEAR SYSTEMS OF EQUATIONS. DIFFERENT METHODS

2.1. SUBSTITUTION METHOD

Let's revise it:

1. Work out the value of an unknown in one of the equations.
2. Substitute the expression of this unknown into the other equation, obtaining an equation with one unknown.
3. Solve the equation.
4. The value obtained is substituted into the other equation.
5. The two values obtained are the solution of the system.

Solved example 1:
$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases}$$

1. Work out the value of x :

$$2x = 16 - 4y \quad x = 8 - 2y$$

2. Substitute the value of x into the other equation:

$$3(8 - 2y) - 4y = -6$$

3. Solve the equation obtained:

$$24 - 6y - 4y = -6 \quad -10y = -30 \quad y = 3$$



4. Substitute the value obtained:

$$x = 8 - 2 \cdot 3 = 8 - 6 \qquad x = 2$$

5. Solution:

$$x = 2, y = 3$$

Example 2:

$$\begin{cases} \frac{x + 3y}{2} = 5 \\ 4 - \frac{2x - y}{2} = 1 \end{cases}$$

2.2. ELIMINATION METHOD

Do you remember it?

1. Prepare the two equations and multiply by the appropriate numbers in order to eliminate one of the unknown values.
2. Add the systems and eliminate one of the unknowns.
3. Solve the resulting equation.
4. Substitute the value obtained into one of the initial equations and then solve it.
5. The two values obtained are the solution of the system.



Solved example 1:
$$\begin{cases} 3x - 4y = -6 \\ 2x + 4y = 16 \end{cases}$$

The easiest method is to remove the y , this way the equations do not have to be prepared. However, by choosing to remove the x , the process can be seen better.

$$\begin{cases} 3x - 4y = -6 & \xrightarrow{\times 2} 6x - 8y = -12 \\ 2x + 4y = 16 & \xrightarrow{\times (-3)} -6x - 12y = -48 \end{cases}$$

Add and solve the equation:

$$\begin{array}{rcl} \begin{cases} \cancel{6x} - 8y = -12 \\ \cancel{-6x} - 12y = -48 \end{cases} & & \\ \hline -20y = -60 & y = 3 & \end{array}$$

Replace the value of y in the second equation.

$$2x + 4 \cdot 3 = 16 \quad 2x + 12 = 16 \quad 2x = 4 \quad x = 2$$

Solution:

$$x = 2, y = 3$$

Example 2:
$$\begin{cases} \frac{x+1}{3} + \frac{y-1}{2} = 0 \\ \frac{x+2y}{3} - \frac{x+y+2}{4} = 0 \end{cases}$$



2.3. "EQUALIZATION" METHOD

Here are the steps to remember what this method consists in:

1. Work out the value of the same variable in both equations (you choose which variable).
2. Make both expressions equal.
3. Solve this equation.
4. To calculate the value of the other variable, plug the first back into one of the equations, "substituting".

Example: Solve using the Equalization Method:

$$\begin{cases} 3x + y = 5 \\ -4x + y = -9 \end{cases}$$

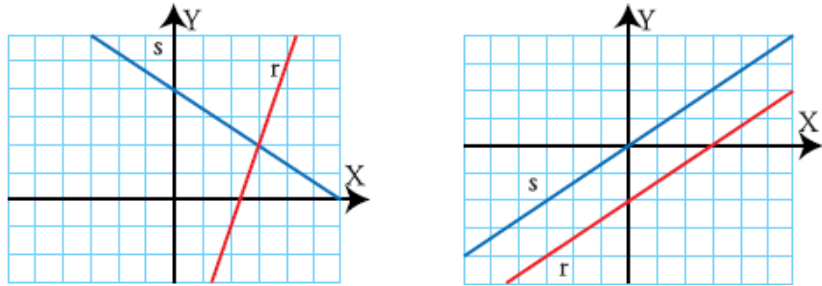
Solution:



2.4. GRAPHICAL METHOD

Let's revise it:

- In which point do these straight lines cross?
- Have they got any point in common?
- How are these straight lines?



The **graphical method** consists in graphing every equation in the system and then using the graph to find the coordinates of the point or points where the graphs intersect. **The point of intersection is the solution.**

Solved example: Use the graphical method to solve the system:

$$\begin{cases} x + y = 4 \\ 2x - y = -1 \end{cases}$$

1st Step: Draw the first straight line corresponding to the first equation.

$$x + y = 4 \Rightarrow y = 4 - x$$

x	y = 4 - x
-2	6
-1	5
0	4
1	3
2	2

2nd Step: Draw the second straight line corresponding to the second equation.

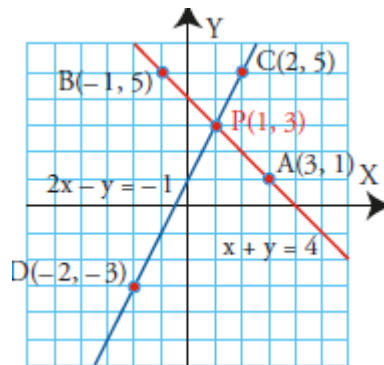
$$2x - y = -1 \Rightarrow y = 2x + 1$$

x	y = 2x + 1
-2	-3
-1	-1
0	1
1	3
2	5



3rd Step: The solution is the point of intersection of these straight lines:

$$x = 1, y = 3.$$



Example: Use the graphical method to solve the following system of equations:

$$\begin{cases} x - y = -2 \\ -2x + y = 3 \end{cases}$$

Note: Graph both equations very precisely. If you don't graph carefully, your point of intersection will be way off.

Example: Use the graphic method to solve the following system of linear equations:

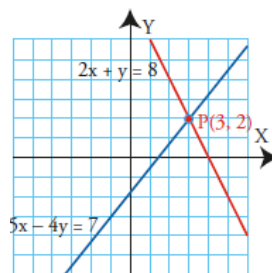
$$\begin{cases} 2x + y = 4 \\ -x + y = 1 \end{cases}$$



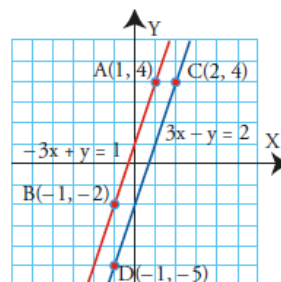
2.5. NUMBER OF SOLUTIONS IN A SYSTEM OF EQUATIONS

Systems of equations can be classified according to the number of solutions they have:

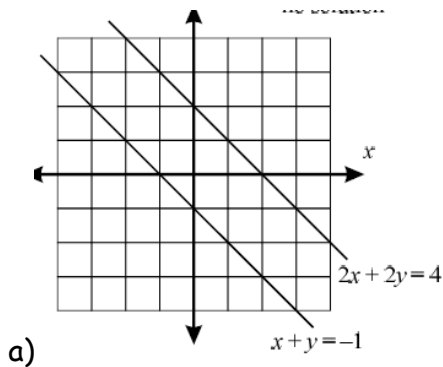
- **Compatible system:** if the system has got one solution, then the straight lines cut across each other at one point. Example:



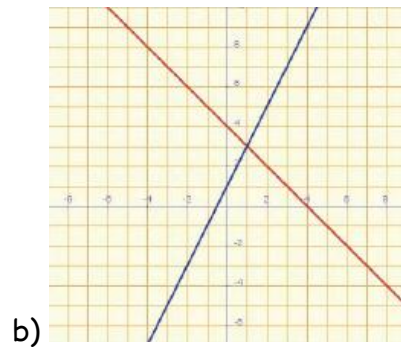
- **Incompatible system:** if the system hasn't got a solution, then the straight lines are parallel. Example:



Example: Look at these graphics and write the solutions of these systems of equations:



SOLUTION: _____



SOLUTION: _____

TO REVISE THIS THEORY, YOU CAN VISIT THIS USEFUL AND ENJOYABLE WEBSITE, WHERE YOU CAN WATCH SOME VIDEOS. COME ON, GIVE IT A TRY!



<http://www.math-videos-online.com/systems-of-equations.html>

Exercise 1. Solve the following system using the graphic method. Decide if it is compatible or incompatible:

$$\begin{cases} 2x + 3y = -1 \\ 3x + 4y = 0 \end{cases}$$

Exercise 2. Solve the following systems using the most appropriate method. Decide if they are compatible or incompatible:

a)
$$\left. \begin{aligned} \frac{x}{2} - \frac{x+y}{6} &= \frac{11}{6} \\ \frac{2x-3y}{5} - \frac{1}{10} &= \frac{33}{10} \end{aligned} \right\}$$

b)
$$\left. \begin{aligned} 2x + 4y &= 7 \\ \frac{x}{3} - \frac{2x-5y}{6} &= \frac{5}{4} \end{aligned} \right\}$$



$$\text{c) } \left. \begin{array}{l} 2x - \frac{3x - y}{5} = \frac{22}{5} \\ \frac{y}{3} + \frac{4x - 3y}{4} = \frac{31}{12} \end{array} \right\}$$

$$\text{d) } \left. \begin{array}{l} \frac{x}{2} - \frac{x - y}{3} = \frac{1}{6} \\ \frac{1}{4} + y - \frac{2x - 5y}{6} = \frac{19}{12} \end{array} \right\}$$



3. NONLINEAR SYSTEMS OF EQUATIONS

A system of equations is nonlinear when at least one of its equations is not linear (i.e., it does not represent a straight line in a graph).

Example 1:

$$\begin{cases} x^2 + y^2 = 25 \\ x + y = 7 \end{cases}$$

The solution of these systems is usually found by means of the **substitution method**. To use this method, follow these steps:

1. Work out the value of an unknown in one of the equations, preferably in an equation of the first grade.

$$y = 7 - x$$

2. Substitute the value of the unknown in the other equation.



$$x^2 + (7 - x)^2 = 25$$

3. Solve the resulting equation.

$$x^2 + 49 - 14x + x^2 = 25$$

$$2x^2 - 14x + 24 = 0$$

$$x^2 - 7x + 12 = 0$$

4. Each of the values obtained are substituted into the other equation, and corresponding values of the other unknown are obtained.

$$x = 3 \quad y = 7 - 3 \quad y = 4$$

$$x = 4 \quad y = 7 - 4 \quad y = 3$$

Example 2: Solve the following system of equations:

$$\left. \begin{array}{l} y = x^2 - 6x + 7 \\ y = x - 3 \end{array} \right\}$$

Example 3: Solve the following system of equations:

$$\left. \begin{array}{l} x^2 + y^2 - 4x - 6y + 11 = 0 \\ x^2 + y^2 - 6x - 8y + 21 = 0 \end{array} \right\}$$



Exercise 3. Solve the following nonlinear systems of equations, and think about their graphical interpretation:

a)
$$\left. \begin{array}{l} y = x^2 + 4x - 1 \\ y = 2x + 2 \end{array} \right\}$$

b)
$$\left. \begin{array}{l} x^2 + y^2 - 4x - 2y = 20 \\ x^2 + y^2 - 12x + 2y = -12 \end{array} \right\}$$



$$\text{c) } \left. \begin{array}{l} xy = 6 \\ 3x - 2y = 0 \end{array} \right\}$$

$$\text{d) } \left. \begin{array}{l} xy = 4 \\ x^2 + y^2 = 17 \end{array} \right\}$$



Exercise 4. Solve the following nonlinear systems of equations and interpret their solutions graphically:

a)
$$\left. \begin{array}{l} y = -x^2 + 4x + 1 \\ x + y = 5 \end{array} \right\}$$

b)
$$\left. \begin{array}{l} x^2 + y^2 = 18 \\ x^2 + y^2 - 4x - 4y + 6 = 0 \end{array} \right\}$$



$$\text{c) } \left. \begin{array}{l} x - 3y = -5 \\ xy - 2x - y = 1 \end{array} \right\}$$

$$\text{d) } \left. \begin{array}{l} xy = 3 \\ x^2 + y^2 - 4x - 4y + 6 = 0 \end{array} \right\}$$



4. EXPONENTIAL AND LOGARITHMIC SYSTEMS OF EQUATIONS

4.1. EXPONENTIAL SYSTEMS

To solve an exponential system we must make two changes of variables to convert it into a linear system. Look at the following **example**:

$$\begin{cases} 3^x + 7^y = 10 \\ 2 \cdot 3^x - 5 \cdot 7^y = 13 \end{cases}$$

4.2. LOGARITHMIC SYSTEMS

To solve a logarithm system we use the same method that we use to solve linear systems. The system will convert it into logarithm equations that we will have to work out.

Study this **example**:

$$\begin{cases} 2 \log x + \log y = \log 9 \\ \log x - \log y = \log 3 \end{cases}$$



Exercise 5. Solve the following systems of equations:

a)
$$\begin{cases} 2^x + 3^y = 7 \\ 2^x - 3^y = 1 \end{cases}$$

b)
$$\begin{cases} \log x + \log y = \log 12 \\ \log x - \log y = \log 3 \end{cases}$$

c)
$$\begin{cases} 3^x + 5^y = 28 \\ 8 \cdot 3^x - 5^y = -1 \end{cases}$$

$$\text{d) } \left. \begin{array}{l} \log (x - 1) - \log (y + 3) = 0 \\ 2 \log x + \log (y + 1) = 4 \log 2 \end{array} \right\}$$

Exercise 6. Solve the following systems of equations:

$$\text{a) } \left. \begin{array}{l} 2^x + 3^y = 17 \\ 5 \cdot 2^x - 4 \cdot 3^y = 4 \end{array} \right\}$$

$$\text{b) } \left. \begin{array}{l} \log (x + 1) + \log y = 2 \log 2 \\ 2 \log x + \log y = \log 2 \end{array} \right\}$$



$$\text{c) } \left. \begin{array}{l} 7^x - 5^y = 338 \\ 2 \cdot 7^x - 3 \cdot 5^y = 671 \end{array} \right\}$$

$$\text{d) } \left. \begin{array}{l} \log x - \log y = \log 3 \\ 2 \log x - 3 \log y = \log 3 \end{array} \right\}$$

$$\text{e) } \left. \begin{array}{l} y = e^{x+2} \\ y = e^{-x} \end{array} \right\}$$

