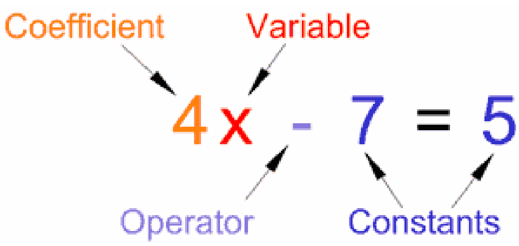
UNIT 4. EQUATIONS

1. EQUATIONS AND SOLVING EQUATIONS

An equation is a mathematical statement that has an expression on the left side of the equals sign (=) with the same value as the expression on the right side. One of the terms in an equation may not be known, and needs to be determined. Often this unknown term is represented by a letter such as "x".

In order that people can talk about equations, there are names for all the different parts (it's better than saying: "this thing here"!)

Here is an equation that says 4x-7 equals 5, and all its parts:

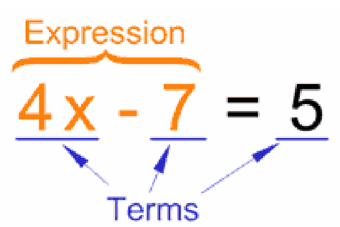


A **variable** is a symbol for a number we don't know yet. It is usually a letter like x or y.

A number on its own is called a **constant**.

A **coefficient** is a number used to multiply a variable (4x means 4 times x, so 4 is a coefficient).

An **operator** is a symbol (such as +, ×, etc) that represents an operation.



A **term** is either a single constant or a variable, or constants and variables multiplied together.

An **expression** is a group of terms (the terms are separated by operators, such as + or - signs).

So, now we can say things like "that expression has only two terms", or "the second term is a constant", or "are you sure the coefficient is really 4?"

Solving an equation means manipulating the expressions and finding the value of the unknown variables. In order to find out the unknown variable, you must isolate it. To avoid altering an equation while we are manipulating it, we must do exactly the same thing to each side of the equation. If we add (or subtract) a quantity from one side, we must add (or subtract) that same quantity from the other side.

2. FIRST AND SECOND ORDER EQUATIONS

2.1. FIRST ORDER EQUATIONS

A **first order equation with an unknown number** is an equation with only one unknown number, and the biggest index in the variable is one. The general expression of a first order equation is:

$$ax + b = 0 \quad (a \neq 0)$$

This equation has only one solution: $x = -b/a$

To solve a first order equation it is recommended to follow the following steps:

1. Rewrite the equation without parentheses by using the distributive property.	$x(2 - 3) = 3(x - {}^{3}/_{2})$ $2x - 3x = 3x - {}^{9}/_{2}$
2. Rewrite it without fractions by multiplying both sides by the least common denominator.	$4x - 6x = 6x - 9$
3. Collect variables on one side and constants on the other.	$4x - 6x - 6x = -9$
4. Combine like terms.	$-8x = -9$
5. Divide by the coefficient of the variable.	$x = {}^{9}/_{8}$

2.2. SECOND ORDER EQUATIONS

2.2.1. COMPLETE QUADRATIC EQUATION

The Quadratic Formula uses the "a", "b", and "c" from "ax2 + bx + c", where "a", "b", and "c" are just numbers; they are the "numerical coefficients". In a complete quadratic equation the numbers a, b and c are not zero.

The Formula is derived from the process of completing the square, and it is formally stated as:

For $ax^2 + bx + c = 0$, the value of x is given by:	$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

DON'T FORGET:

For the Quadratic Formula to work, you must have your equation arranged in the form "(quadratic) = 0". Also, the "2a" in the denominator of the Formula is underneath everything above, not just the square root. And it's a "2a" under there, not just a plain "2". Make sure that you are careful not to forget the square root or the "plus/minus" in the middle of your calculations, or I can guarantee that you will forget to "put them back" on your test, and you'll be in trouble. Remember that "b^2" means "the square of ALL of b, including its sign", so don't leave b^2 being negative, even if b is negative, because the square of a negative is a positive.

In other words, don't be in a hurry and pay attention to all the details.

Here are some **examples** of how the Quadratic Formula works:

EXAMPLE 1: Solve $x^2 + 3x - 4 = 0$

This quadratic can be factored this way:

$x^2 + 3x - 4 = (x + 4)(x - 1) = 0$

...so we already know that the solutions to this equation are $x = -4$ and $x = 1$. How would our solution look in the Quadratic Formula? Using $a = 1$, $b = 3$, and $c = -4$, our solution looks like this:

$$\begin{aligned} x &= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} \\ &= \frac{-3 \pm 5}{2} = \frac{-3-5}{2}, \ \frac{-3+5}{2} \\ &= \frac{-8}{2}, \ \frac{2}{2} = -4, \ 1 = x \end{aligned}$$

Then, as expected, the solution is **$x = -4$, $x = 1$**

CARDANO-VIETA'S FORMULA:

If you consider the quadratic equation as $x^2 - Sx + P = 0$, the solutions of this equation are two numbers x_1 and x_2, which satisfy these conditions:

$$x_1 + x_2 = S$$
$$x_1 \cdot x_2 = P$$

EXAMPLE: Find out the solutions of the following equations: $x^2 - 5x + 6 = 0$

The solutions of the equations are $x_1=2$ and $x_2=3$, because 3+2=5 and 2·3=6.

Now it is your turn: solve the following equations using Cardano-Vieta's Formula:

a) $x^2 - 7x + 10 = 0$

b) $x^2 - x - 6 = 0$

c) $x^2 + 8x + 16 = 0$

2.2.2. INCOMPLETE QUADRATIC EQUATION

If the coefficient "b" or "c" is zero, we have got an incomplete quadratic equation. We can always use the Quadratic Formula and it works, but in these cases, there is an easier method to solve it. For example:

COMPLETE CUADRATIC EQUATION	INCOMPLETE CUADRATIC EQUATIONS
$x^2 - 5x + 6 = 0$	$2x^2 - 7x = 0$ $5x^2 - 4 = 0$

Continue reading to know the new method in each case.

Solving a quadratic equation when c=0, $ax^2 + bx = 0$.

You have to extract common factor and one solution will be 0.

Example: Solve $3x^2 + 4x = 0$. Extracting common factor:

$$x(3x+4) = 0 \Rightarrow \begin{cases} x = 0 \\ 3x + 4 = 0 \Rightarrow 3x = -4 \Rightarrow x = -4/3 \end{cases}$$

So, the solutions are x= 0 and x = -4/3.

Solving a quadratic equation when b = 0, $ax^2 + c = 0$.

You have to find the value of x^2 and calculate the square root.

Example: Solve $4x^2 - 1 = 0$.

$$4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

So, the solutions are $x_1 = -\frac{1}{2}, x_2 = \frac{1}{2}$.

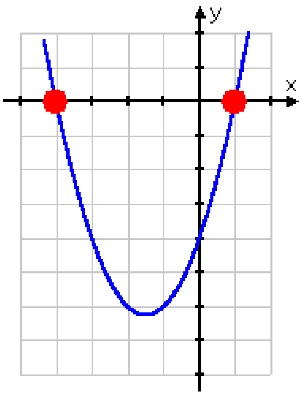
Special case: Would you know what the solutions are if the quadratic equation is $ax^2 = 0$?

2.2.3. CORRESPONDENCE BETWEEN THE SOLUTIONS OF AN EQUATION AND A GRAPH.

Suppose you have $ax^2 + bx + c = y$, and you have to find values of x for $y = 0$.

The corresponding x values are the x-intercepts of the graph (the x-intercepts are the points where the graph crosses the x-axis).

So solving $ax^2 + bx + c = 0$ for x means that you are trying to find x-intercepts. Since there were two solutions for $x^2 + 3x - 4 = 0$, there must be two x-intercepts on the graph. Graphing, we get the curve below:



As you can see, the x-intercepts match the solutions, crossing the x-axis at $x = -4$ and $x = 1$.

This shows the connection between graphing and solving: When you are solving "(quadratic) = 0", you are finding the x-intercepts of the graph. This can be useful if you have a graphing calculator, because you can use the Quadratic Formula (when necessary) to solve a quadratic, and then use your graphing calculator to make sure that the displayed x-intercepts have the same values as the solutions provided by the Quadratic Formula.

Remember: The "solutions" of an equation are also the x-intercepts of the corresponding graph.

EXAMPLE 2: Solve $x(x - 2) = 4$. Round your answer to two decimal places.

There is a problem: I cannot apply the Quadratic Formula at this point, and I cannot factor it either, because on the right we have 4, not 0, so we cannot claim that "$x = 2$, $x - 2 = 2$". This is *not* how "solving by factoring" works.

I *must* first rearrange the equation in the form "(quadratic) = 0", whether I'm factoring or using the Quadratic Formula. First I have to do the multiplication on the left-hand side, and then I have to move the 4 over:

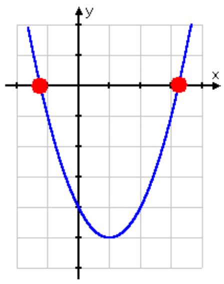
$$x(x - 2) = 4$$
$$x^2 - 2x = 4$$
$$x^2 - 2x - 4 = 0$$

Now we can use the Quadratic Formula; in this case, a = 1, b = –2, and c = –4:

$$x = \frac{-(-2) \pm \sqrt{(-2)^2 - 4(1)(-4)}}{2(1)}$$
$$= \frac{2 \pm \sqrt{4+16}}{2} = \frac{2 \pm \sqrt{20}}{2} = \frac{2 \pm \sqrt{4}\sqrt{5}}{2}$$
$$= \frac{2 \pm 2\sqrt{5}}{2} = \frac{2(1 \pm \sqrt{5})}{2(1)} = 1 \pm \sqrt{5}$$
$$\approx -1.236068,\ \ 3.236968 = x$$

Then the answer is: x = –1.24, x = 3.24, rounded to two places.

Here's what the graph looks like:



EXAMPLE 3. Solve $9x^2 + 12x + 4 = 0$.

Using a = 9, b = 12 and c = 4, the Quadratic Formula gives:

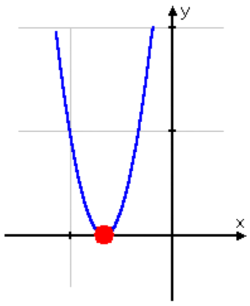
$$x = \frac{-(12) \pm \sqrt{(12)^2 - 4(9)(4)}}{2(9)}$$
$$= \frac{-12 \pm \sqrt{144 - 144}}{18} = \frac{-12 \pm \sqrt{0}}{18}$$
$$= \frac{-12 \pm 0}{18} = \frac{-12}{18} = \frac{6(-2)}{6(3)} = -\frac{2}{3} = x$$

Then the answer is $x = {}^{-2}/_{3}$

In the previous examples, I had obtained two solutions because of the "plus-minus" part of the Formula. In this case, though, the square root was reduced to zero, so the plus-minus was not important.

This solution is called a "repeated" root.

This is what the graph looks like:



EXAMPLE 4. Solve $3x^2 + 4x + 2 = 0$.

The Quadratic Formula always works; in this case, $a = 3$, $b = 4$, and $c = 2$:

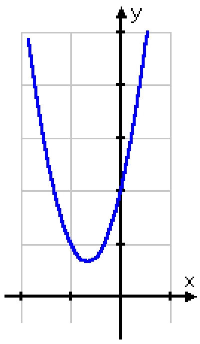
$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(2)}}{2(3)}$$

$$= \frac{-4 \pm \sqrt{16-24}}{6} = \frac{-4 \pm \sqrt{-8}}{6} = x$$

At this point, I have a negative number inside the square root. If you haven't learned about complex numbers yet, then you will have to stop here, and the answer will be "no solution", but, just to finish our example, we can continue the calculations:

$$\frac{-4 \pm \sqrt{-8}}{6} = \frac{-4 \pm 2\sqrt{-2}}{6} = \frac{-2 \pm \sqrt{2}i}{3} = -\frac{2}{3} \pm \frac{\sqrt{2}}{3}i = x$$

Here's the graph:



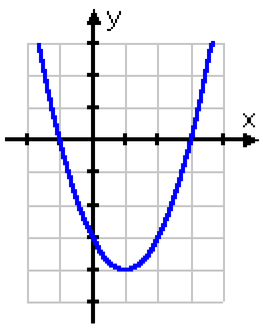
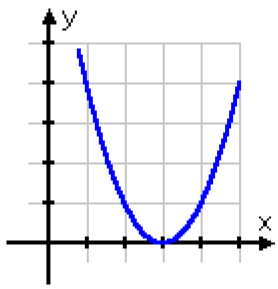
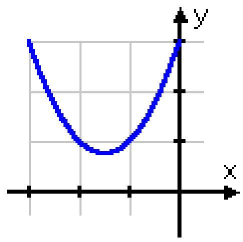
This is always true: if you get a negative value inside the square root, then there will be no *real number* solution, and therefore no x-intercepts.

Do you know what a complex number is? If you are curious to know about complex numbers, you can read the following link:

http://www.purplemath.com/modules/complex.htm

SUMMING UP:

This relationship between the value inside the square root (**the discriminant:** Δ), the type of solutions (two different solutions, one repeated solution, or no real solutions), and the number of x-intercepts (on the corresponding graph) of the quadratic is summarized in this table:

$x^2 - 2x - 3=0$	$x^2 - 6x + 9=0$	$x^2 + 3x + 3=0$
$x=\frac{2\pm\sqrt{(-2)^2-4(-3)}}{2}$ $=\frac{2\pm\sqrt{4+12}}{2}$ $=\frac{2\pm\sqrt{16}}{2}=\frac{2\pm 4}{2}$ $=\frac{-2}{2},\ \frac{6}{2}=-1,\ 3$	$x=\frac{6\pm\sqrt{(-6)^2-4(9)}}{2}$ $=\frac{6\pm\sqrt{36-36}}{2}$ $=\frac{6\pm\sqrt{0}}{2}=\frac{6\pm 0}{2}=3$	$x=\frac{-3\pm\sqrt{(3)^2-4(3)}}{2}$ $=\frac{-3\pm\sqrt{9-12}}{2}$ $=\frac{-3\pm\sqrt{-3}}{2}$ $=-\frac{3}{2}\pm\frac{\sqrt{3}i}{2}$
a positive number inside the square root $\Delta > 0$	zero inside the square root $\Delta = 0$	a negative number inside the square root $\Delta < 0$
two real solutions	one (repeated) real solution	two complex solutions
		
two distinct x-intercepts	one (repeated) x-intercept	no x-intercepts

Therefore, studying the sign of the discriminant we will know how many solutions a quadratic equation has got.

The discriminant is: $\Delta = b^2 - 4ac$.

2.2.4. FACTORIAL DECOMPOSITION OF A SECOND ORDER POLYNOMIAL

The factorial decomposition of a second order polynomial is:

$$ax^2 + bx + c = a(x - x_1)(x - x_2)$$

Where x_1 and x_2 are the roots of the equation: $ax^2 + bx + c = 0$

3. BIQUADRATIC EQUATIONS

Biquadratic equations are quartic equations (an equation of the fourth degree) with no odd-degree terms (no x or x^3):

$$ax^4 + bx^2 + c = 0$$

To solve biquadratic equations, change $x^2 = t$, $x^4 = t^2$; this generates a quadratic equation with the unknown variable **t**:

$$at^2 + bt + c = 0$$

As $t = x^2$, for every positive value of **t** there are **two values of x**: $x = \pm\sqrt{t}$

Example 1:

$$x^4 - 13x^2 + 36 = 0$$

$$x^2 = t$$

$$t^2 - 13t + 36 = 0$$

$$t = \frac{13 \pm \sqrt{169 - 144}}{2} = \frac{13 \pm 5}{2} = \begin{cases} \nearrow t_1 = \frac{18}{2} = 9 \\ \searrow t_2 = \frac{8}{2} = 4 \end{cases}$$

$$x^2 = 9 \qquad x = \pm\sqrt{9} = \begin{cases} \nearrow x_1 = 3 \\ \searrow x_2 = -3 \end{cases}$$

$$x^2 = 4 \qquad x = \pm\sqrt{4} = \begin{cases} \nearrow x_3 = 2 \\ \searrow x_4 = -2 \end{cases}$$

Example 2:

$x^4 - 10x^2 + 9 = 0 \quad \rightarrow x^2 = t \quad \rightarrow t^2 - 10t + 9 = 0$

$$t = \frac{10 \pm \sqrt{10^2 - 4 \cdot 9}}{2} = \frac{10 \pm \sqrt{100 - 36}}{2} = \frac{10 \pm \sqrt{64}}{2} = \frac{10 \pm 8}{2} = \begin{cases} \nearrow t_1 = \frac{18}{2} = 9 \\ \searrow t_2 = \frac{2}{2} = 1 \end{cases}$$

$x^2 = 9 \qquad x = \pm\sqrt{9} = \pm 3$

$x^2 = 1 \qquad x = \pm\sqrt{1} = \pm 1$

The same procedure can be used to solve the equations of the type:

$ax^6 + bx^3 + c = 0$ or $ax^8 + bx^4 + c = 0$ or $ax^{10} + bx^5 + c = 0$

Example 3: $x^6 - 7x^3 + 6 = 0$

$x^3 = t$

$t^2 - 7t + 6 = 0$

$$t = \frac{7 \pm \sqrt{49 - 24}}{2} = \frac{7 \pm 5}{2} = \begin{cases} \nearrow t_1 = \frac{12}{2} = 6 \\ \searrow t_2 = \frac{2}{2} = 1 \end{cases}$$

$x^3 = 6 \qquad x = \sqrt[3]{6}$

$x^3 = 1 \qquad x = \sqrt[3]{1} \qquad x = 1$

4. RATIONAL EQUATIONS

Rational polynomial equations are of the form $\frac{P(x)}{Q(x)}=0$, where P(x) and Q(x) are polynomials.

To solve rational equations multiply both sides of the equation by the least common multiple of the denominators.

Example 1:

$$\frac{1}{x-2}+\frac{1}{x+2}=\frac{1}{x^2-4}$$

$$x^2-4=(x-2)\cdot(x+2)$$

$$lcm\left(x-2, x+2, x^2-4\right)=(x-2)\cdot(x+2)$$

$$x+2+x-2=1 \qquad 2x=1 \qquad x=\frac{1}{2}$$

Don't forget to verify the solution that you have found:

$$\frac{1}{\frac{1}{2}-2}+\frac{1}{\frac{1}{2}+2}=\frac{1}{\left(\frac{1}{2}\right)^2-4}$$

$$\frac{1}{-\frac{3}{2}}+\frac{1}{\frac{5}{2}}=\frac{1}{\frac{-15}{4}} \qquad -\frac{2}{3}+\frac{2}{5}=-\frac{4}{15} \qquad -\frac{4}{15}=-\frac{4}{15}$$

The solution is: $\boxed{x=\frac{1}{2}}$

Example 2: $\frac{1}{x^2-x}-\frac{1}{x-1}=0$

<u>Example 3:</u> $\frac{3}{x} = 1 + \frac{x-13}{6}$

5. RADICAL EQUATIONS OR IRRATIONAL EQUATIONS

Irrational equations, or **radical equations**, are those with the unknown value under the radical sign.

$$\sqrt{2x-3}-x=-1$$

To solve radical equations, follow these steps:

1. Isolate a radical in one of the two members and pass the other terms to the other member.
2. Square both members.
3. Solve the equation obtained.
4. **Check if the solutions obtained verify the initial equation**. Keep in mind that squaring an equation can only be done if another has the same solution as those that are given.
5. If the equation has several radicals, repeat the first two phases of the process to remove all of them.

Example 1: Solve the equation $\sqrt{2x-3}-x=-1$

1. Isolate the radical: $\sqrt{2x-3}=-1+x$

2. Square both members: $\left(\sqrt{2x-3}\right)^2=(-1+x)^2$ → $2x-3=1-2x+x^2$

3. Solve the equation: $x^2-4x+4=0$ → $x=\dfrac{4\pm\sqrt{16-16}}{2}=\dfrac{4}{2}=2$

4. Verify:

$$\sqrt{2\cdot2-3}-2=-1 \qquad 1-2=-1$$

The equation has the solution **x = 2**.

Example 2: $\sqrt{x} + \sqrt{x-4} = 2 \rightarrow \sqrt{x} = 2 - \sqrt{x-4} \rightarrow$ $\left(\sqrt{x}\right)^2 = \left(2 - \sqrt{x-4}\right)^2$

$x = 4 - 4\sqrt{x-4} + x - 4 \rightarrow 4\sqrt{x-4} = 0 \quad \sqrt{x-4} = 0$

$\left(\sqrt{x-4}\right)^2 = 0^2 \quad x - 4 = 0 \quad x = 4$

Verifying the solution: $\sqrt{4} + \sqrt{4-4} = 2 \quad 2 + 0 = 2$

So, the equation has the solution **x = 4**.

Example 3: $\sqrt{5x+4} - 1 = 2x$

Example 4: $3\sqrt{x-1} + 11 = 2x$

Example 5: $\sqrt{2x-1}+\sqrt{x+4}=6$

6. EXPONENTIAL EQUATIONS

An **exponential equation** is one in which a variable occurs in the exponent, for example: $3^x = 4x - 1$

1st case: If you can express both sides of the equation as powers of the same base, you can set the exponents equal to solve for x.

Examples:

	Solve for *x*.	Answer
1.	$7^{2x+1} = 7^{3x-2}$	Since the bases are the same, set the exponents equal to one another: $2x + 1 = 3x - 2$ $3 = x$
2.	$3^{2x-1} = 27^x$	27 can be expressed as a power of 3: $3^{2x-1} = \left(3^3\right)^x = 3^{3x}$, so $2x - 1 = 3x$ $-1 = x$
3.	$5^{3x-8} = 25^{2x}$	25 can be expressed as a power of 5: $5^{3x-8} = \left(5^2\right)^{2x} = 5^{4x}$ $3x - 8 = 4x$ $-8 = x$

2nd Case: Unfortunately, not all exponential equations can be expressed in terms of a common base. If it is possible, we can reduce it into a second degree equation (quadratic equation):

Example 1:

$$9^x - 12\cdot 3^x + 27 = 0$$

This question requires some additional thinking. Because of the differing powers of 3, our previous methods will not be of much help. We will need a different strategy with this problem.

Firstly, as $9=3^2$ we can re-write the equation as $\left(3^2\right)^x - 12\cdot 3^x + 27 = 0$

Secondly, change **3^x=z;** this generates the following quadratic equation:

$z^2 - 12\cdot z + 27 = 0$.

The solutions of this equation are z=3 and z=9.

Finally, you have to obtain the solutions for "x":

As 3^x = z = 3, then **x=1**. Also, as 3^x = z = 9, then **x=2**.

Example 2: $4^x - 10 \cdot 2^x + 16 = 0$

3rd Case: Unfortunately, not all exponential equations can be expressed in terms of a common base or reduced into a quadratic equation. If we are in this situation, for these equations, logarithms are used to arrive at a solution (you may solve them using common log or natural ln.).

Remember:	
$\log a^r = r \log a$	$\ln 1 = 0$
$\log_b 1 = 0$	$\ln e = 1$
$\log_b b = 1$	$\ln e^x = x$
$\log_b b^x = x$	

To solve most exponential equations:
1. Isolate the exponential expression.
2. Take log or ln of both sides.
3. Solve for the variable.

Here you have a lot of solved examples, check them:

Example	Solve for *x*, *to the nearest thousandth.*	Answer
1.	$5^x = 7$	$\log 5^x = \log 7$ $x\log 5 = \log 7$ $x = \dfrac{\log 7}{\log 5} \approx 1.209$ • Take the log of both sides. • Apply the log rule for exponents shown above. • Solve for x. • Estimate answer using calculator.
2.	$3^{2x+1} = 15$	$\log 3^{2x+1} = \log 15$ $(2x+1)\log 3 = \log 15$ $2x+1 = \dfrac{\log 15}{\log 3}$ $2x = \dfrac{\log 15}{\log 3} - 1$ $x = \dfrac{\dfrac{\log 15}{\log 3} - 1}{2} \approx .732$ **OR** $\ln 3^{2x+1} = \ln 15$ $(2x+1)\ln 3 = \ln 15$ $2x+1 = \dfrac{\ln 15}{\ln 3}$ $2x = \dfrac{\ln 15}{\ln 3} - 1$ $x = \dfrac{\dfrac{\ln 15}{\ln 3} - 1}{2} \approx .732$ Also, log base 3 can be used.
3.	$\left(\dfrac{1}{2}\right)^x = 3$	$\log\left(\dfrac{1}{2}\right)^x = \log 3$ $x\log\left(\dfrac{1}{2}\right) = \log 3$ $x = \dfrac{\log 3}{\log\left(\dfrac{1}{2}\right)} \approx -1.585$ Also, log base 1/2 can be used as a solution method.
4.	$5^x = 13$	$\log 5^x = \log 13$ $x\log 5 = \log 13$ $x = \dfrac{\log 13}{\log 5} \approx 1.594$

Example	Solve for *x*, *to the nearest thousandth.*	Answer
5.	$e^x = 30$	Since the natural log is the inverse of the exponential function, use **ln** to quickly solve this problem. $\ln e^x = \ln 30$ $x = \ln 30 \approx 3.401$
6.	$150e^{0.05x} = 340$	$150e^{0.05x} = 340$ $e^{0.05x} = \frac{340}{150}$ $\ln e^{0.05x} = \ln\frac{34}{15}$ $0.05x = \ln\frac{34}{15}$ $x = \frac{\ln\frac{34}{15}}{0.05} \approx 16.366$ • First, get rid of the coefficient of the exponential term (divide by 150). • Now, proceed using **ln** to solve quickly. • Do not round too quickly. Be sure to carry enough decimal values to allow you to round to thousandths (in this case) for the final answer.
7.	$4^{x+2} - 2 = 12$	$4^{x+2} = 14$ • Isolate the exponential $\log 4^{x+2} = \log 14$ • Take the log of both sides $(x+2)\log 4 = \log 14$ • Apply the log property $x + 2 = \frac{\log 14}{\log 4}$ • Divide by log 4 $x = \frac{\log 14}{\log 4} - 2 \approx -0.963225$ • Estimate using calculator
8.	$8 + 2e^x = 12$	$2e^x = 4$ •Isolate the exponential $e^x = 2$ • Divide each side by the coefficient of 2 $\ln e^x = \ln 2$ • Take ln of both sides $x = \ln 2$ • Remember that ln *x* and e^x are inverse functions.

Example	Solve for *x*, *to the nearest thousandth.*	Answer	
9.	$6(2)^{x+3}=30$	$6(2)^{x+3}=30$ $(2)^{x+3}=5$ $\log(2)^{x+3}=\log 5$ $(x+3)\log 2=\log 5$ $x+3=\dfrac{\log 5}{\log 2}$ $x=\dfrac{\log 5}{\log 2}-3\approx -0.6780719($	• First, divide by the coefficient to isolate the exponential • Proceed as shown above.

7. LOGARITHMIC EQUATIONS

Solving logarithmic equations usually requires using the properties of logarithms. The reason you usually need to apply these properties is so that you will have a single logarithmic expression on one or both sides of the equation.

Once you have used the properties of logarithms to condense any log expressions in the equation, you can solve the problem by changing the logarithmic equation into an exponential equation and solving.

EXAMPLE 1: Solve $\log(x-3)+\log x=1$

As mentioned above, the first step used in solving logarithmic equations is to make use of the properties of logs. In this case, we can combine the two log expressions on the left side of the equation into one expression using multiplication.

$$\log(x-3)+\log x=\log((x-3)(x))$$

This means we are now solving the equation

$$\log((x-3)(x))=1$$

$$\log(x^2-3x)=1$$

Now that we have a single log expression equal to a number, we can change the equation into its exponential form (to do this we need to remember that when a base is not given in a problem, it is understood to be the common logarithm with a base of 10).

So our logarithmic equation becomes

$$\log_{10}(x^2 - 3x) = 1$$

$$10^1 = (x^2 - 3x)$$

Now we need to solve for x. We have to solve a quadratic equation by factoring (most of the time solving by factoring will be enough; you will not need to solve a quadratic by another method very often).

So let's solve for x.

$$10^1 = (x^2 - 3x)$$

$$10 = (x^2 - 3x)$$

$$x^2 - 3x = 10$$

$$x^2 - 3x - 10 = 0$$

Factoring and setting each term equal to zero results in:

$(x - 5)(x + 2) = 0$

$x - 5 = 0$ or $x + 2 = 0$

$x = 5$ or $x = -2$

When solving logarithmic equations, we must ALWAYS check our answers. Logarithmic functions are not defined for negative values. Therefore we have to plug in our answers and make sure we are not taking the log of a negative number.

We will plug in x = 5 into the original equation:

$$\log(5-3) + \log(5) = 1$$

We do not actually have to continue in the checking process as soon as we see that we are not taking the log of a negative number.

Now let's plug in x = -2 into the original equation.

$$\log(-2-3) + \log(-2) = 1$$

As soon as we see that we are taking the log of a negative number we know that x = -2 is NOT a solution.

So our only solution is x = 5.

EXAMPLE 2: Solve $\ln(x-2) + \ln(2x-3) = 2\ln(x)$

This problem is slightly different from the last example that we worked on. First of all, it involves the natural logarithm. But you may also notice that there are log expressions on both sides of the equation. Our approach to this type of problem is to write each side as a single log expression. Once we do that, we can apply the property of logarithms that says

If $\log_b(x) = \log_b(y)$ then x = y.

Let's get started.

First we'll apply properties of logs and write the left side of the equation as a single expression using multiplication, and write the right side with an exponent of 2 rather than a coefficient of 2.

$$\ln(x-2)+\ln(2x-3)=2\ln(x)$$
$$\ln\big((x-2)(2x-3)\big)=\ln(x^2)$$
$$\ln(2x^2-7x+6)=\ln(x^2)$$

Now we use the property of logs shown above to get

$$2x^2-7x+6=x^2$$
$$x^2-7x+6=0$$

Factoring and setting each term equal to zero results in

$$(x-6)(x-1)=0$$

x - 6 = 0 or x - 1 = 0

x = 6 or x = 1

If we check x = 6 in the original equation we do not have to take the log of a negative number, but plugging x = 1 into the original equation will cause us to take the log of a negative number, so it cannot be a solution.

Therefore, our answer to this problem is x = 6.

EXAMPLE 3: Solve the following equation: $\log(5x+3)-\log x=1$

EXAMPLE 4: $\log x + \log 80 = 3$

EXERCISES:

EXERCISE 1: Solve the following basic equations:

a) $4x + 12 = 6x - 8$

b) $6 + 3x = 4 + 7x - 2x$

c) $8x - 2x + 4 = 2x$

d) $4x + 3x - 4 = 3x + 8$

EXERCISE 2: Solve the following equations:

a) $3(x + 2) + 2x = 5x - 2(x - 4)$

b) $4 - 3(2x + 5) = 5 - (x - 3)$

c) $2(x - 3) + 5(x + 2) = 4(x - 1) + 3$

d) $5 - (2x + 4) = 3 - (3x + 2)$

EXERCISE 3. Solve in your mind:

a) $(x - 2)(x + 3) = 0$

b) $(2x + 1)(x - 4)(3x + 5) = 0$

EXERCISE 4. Solve the following equations with denominators.

a) $\frac{x - 3}{4} = \frac{x - 5}{6} + \frac{x - 1}{9}$

b) $\frac{7-x}{2} = \frac{9}{2} + \frac{7x-5}{10}$

c) $\frac{x}{3} + 3x - \frac{x-2}{4} = \frac{1}{4} + x$

d) $\frac{x-1}{2} - \frac{x-2}{3} + \frac{10-3x}{5} = 0$

EXERCISE 5. Solve the following equations:

a) $12 - (7x + 5) = 4 - (5x + 2)$

b) $\dfrac{6x - 1}{2} = \dfrac{x - 1}{3} + \dfrac{4x + 3}{2}$

c) $\dfrac{4 - x}{5} = 2 - \dfrac{3x - 2}{10}$

d) $\dfrac{3x}{2} - 2(x - 3) - \dfrac{x - 2}{4} = 5 + x$

e) $\frac{x-5}{2} - \frac{2x-3}{3} + \frac{10-x}{12} = 0$

EXERCISE 6. Solve in your mind:

a) $x^2 = 25$

b) $x^2 = 0$

c) $x^2 = 49$

d) $5x^2 = 0$

e) $x^2 - 1 = 0$

EXERCISE 7. Solve the following equations:

a) $x^2 - 6x = 0$

b) $x^2 - 16 = 0$

c) $x^2 - 5x + 6 = 0$

d) $x^2 + 5x = 0$

e) $x^2 - 25 = 0$

f) $x^2 - 9x = 0$

g) $x^2 - 4x + 4 = 0$

h) $x^2 + 8x = 0$

i) $4x^2 - 81 = 0$

j) $2x^2 - 3x - 20 = 0$

k) $8x^2 - 2x - 3 = 0$

l) $x(x - 3) = 10$

ll) $(x + 2)(x + 3) = 6$

m) $(2x - 3)^2 = 8x$

n) $2x(x - 3) = 3x(x - 1)$

ñ) $\frac{3x}{2} - \frac{x^2 + x}{2} = \frac{3}{8}$

o) $$\frac{9x-4}{10} - x + \frac{x^2+2}{30} = 1$$

EXERCISE 8. Solve the following equations

a) $x^4 - 25x^2 + 144 = 0$

b) $x^4 - 16x^2 - 225 = 0$

c) $x^4 - 625 = 0$

d) $x^4 - 4x^2 = 0$

e) $x^6 - 8x^3 = 0$

f) $x^6 - 26x^3 - 27 = 0$

EXERCISE 9. Solve the following equations:

1. $\frac{1}{x+3} = \frac{1}{x} - \frac{1}{6}$

2. $\frac{3x+2}{x+1} - 2 = \frac{3}{4}$

3. $\frac{2}{x-1} + \frac{2x-3}{x^2-1} = \frac{7}{3}$

4. $\frac{3x}{x+2} + \frac{x-1}{6} = x - \frac{2}{3}$

5. $x = 2 + \sqrt{x}$

6. $\sqrt{x-1} - x + 7 = 0$

7. $x - \sqrt{25 - x^2} = 1$

8. $\sqrt{2x^2 - 4} - \sqrt{4x - 6} = 0$

9. $\sqrt{2x + 1} + \sqrt{3x + 4} = 7$

EXERCISE 10. Solve the following equations:

33 a) $3^x = 27$ b) $7^{x+1} = 1$

34 a) $5^{x-1} = 25$ b) $2^x = 1/8$

35 a) $\log x = 0$ b) $\log_2 x = 4$

36 a) $\log_x 3 = 1$ b) $L\,x = 1$

EXERCISE 11. Solve the following equations:

104 $4^x + 2^5 = 3 \cdot 2^{x+2}$

105 $2^{5-x^2} = \dfrac{1}{16}$

106 $5^{2x-2} - 6 \cdot 5^x + 125 = 0$

107 $2^x + 2^{x+1} = 3^x + 3^{x-1}$

108 $1 + 9^x = 3^{x+1} + 3^{x-1}$

109 $2^x + \dfrac{1}{2^{x-2}} = 5$

110 $6^{2x} = 1\,296$

111 $3^x + \dfrac{1}{3^{x-1}} = 4$

112 $5^{1-x} + 5^x = 6$

113 $3^x \cdot 9^x = 9^3$

114 $2^{2x+5} - 5 \cdot 4^{2x-1} + 3\,125 = 53$

115 $2^{x-2} + 28 = 2^{x+2} - 2$

116 $3^{x-4} + 5 \cdot 3^x - 3^{x+1} = 163$

117 $9^x = 3^x + 6$

118 $3^x + 3^{x+1} + 3^{x+2} = 117$

119 $2^x = \left(\frac{1}{3}\right)^{x-1}$

120 $5^{x^2+2x} = 1$

121 $e^{x-1} = 2^{x+1}$

122 $3^{3x-2} = 9^{x^2-2}$

123 $\log(x^2 + 3x + 40) = 1 + \log(3x - 1)$

124 $\log x^2 - \log 3 = \log x + \log 5$

125 $\log x + \log(3x + 5) = 2$

126 $2 \log x - \log(x + 24) = 2$

127 $2 \text{ L } x + \text{L } (x^2 + 2) = \text{L } 3$

128 $\log x + \log 4 = \log(x + 1) + \log 3$

129 $2 \log x + \log x^4 = 6$

130 $2 \, L \, x - L \, 5x = L \, 2$

131 $2 \log x = 4 + \log \frac{x}{10}$

132 $3 \log 2x - 2 \log x = \log (4x + 1)$

133 $3 + \log \frac{3x}{2} = 2 \log x$

134 $\log (x - 2) = 1 + \log 2 - \log (x - 3)$

135 $\log x = 1 - \log (7 - x)$

136 $3 \log (6 - x) - \log (72 - x^3) = 0$

137 $\log \sqrt{3x + 1} + \log 5 = 1 + \log \sqrt{2x - 3}$

138 $(x^2 - 5x + 5) \log 5 + \log 20 = \log 4$