Unit 3. POLYNOMIALS AND ALGEBRAIC FRACTIONS.

1. WHAT IS ALGEBRA? WHY STUDY ALGEBRA?

To answer this question, please visit this webpage:

http://math.about.com/od/algebra/a/WhyAlgebra.htm

Algebraic language (or algebra language) is a language that uses numbers, letters and brackets, connected with operations. It transmits information. It is used in maths and other sciences, and it replaces natural language.

Example:

NATURAL LANGUAGE	ALGEBRAIC LANGUAGE
In a garden, last year, there were **several** trees, and this year, we have planted **five** more trees. How many trees are there this year? Answer: **5 more than last year**.	x 5 x + 5

2. POLYNOMIALS.

Polynomials are algebraic expressions that include real numbers and variables. Division and square roots cannot be involved in the variables. The variables can only include addition, subtraction and multiplication.
Polynomials contain more than one term. Polynomials are the sums of monomials.
A **monomial** has one term: $5y$, or $-8x^2$, or 3.
A **binomial** has two terms: $-3x^2 + 2$, or $-2y^2 + 9y$
A **trinomial** has 3 terms: $-3x^2 + 3x + 2$, or $-2y^2 + 9y + 5$
The **degree** of a term is the exponent of its variable: $3x^2$ has a degree of 2.
When the variable does not have an exponent, we always understand that there is a '1'.

Term	Numerical Coefficient
x^2 $-7x$ -6	1 -7 -6

Polynomials are usually written in decreasing order of terms. The term which is written first is the largest term in the polynomial, or the term with the highest exponent. The first term in a polynomial is called the **leading term**. When a term contains an exponent, it tells you the degree of the term.

Here are some examples of three-term polynomials (each part is a term, and x^2 is called the leading term):

$x^2 - 7x - 6$
$6x^2 - 4xy + 2xy$. → These three-term polynomials have a leading term to the second degree. They are called second degree polynomials, and they are also often called trinomials.

$9x^5 - 3x^4 - 2x - 2$ →This four-term polynomial has a leading term to the fifth degree and another term to the fourth degree. It is called a fifth degree polynomial.

$3x^3$ → This is a one-term algebraic expression which is called a monomial.

EXAMPLE: Visit the following webpage to classify algebraic expressions into monomials, binomials, or trinomials.

http://edhelper.com/polynomials1.htm

VERY IMPORTANT: One thing you will do when solving polynomials is combine like terms (= *términos semejantes*).

The numeric value of an algebraic expression is the value that we get after replacing the variable with a number and doing the operations.

Example: Calculate the numeric value of the algebraic expression $5x + 3$, when $x = 2$:

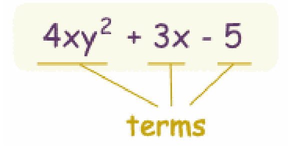
$$5 \cdot 2 + 3 = 13$$

The numeric value is **13**.

3. OPERATIONS WITH POLYNOMIALS.

3.1. ADDING AND SUBTRACTING POLYNOMIALS.

A polynomial looks like this:



Example of a polynomial;
this one has 3 terms.
What is its degree?

To **add polynomials** you simply add all the like terms together... do you remember what *like terms* are?

Like Terms are terms whose variables (and their exponents such as the "2" in x^2) are the same.

In other words, terms that are "like" each other.

Note: the coefficients (the numbers by which you multiply, such as "5" in 5x) can be different.

Example: $7x$ and x and $-2x$ are all like terms because the variables are all x.

ADDING or SUBTRACTING POLYNOMIALS:

Do it in two steps:

1st Place like terms together

2nd Add or subtract the like terms

Here is an animation to show an example:

http://www.mathsisfun.com/algebra/polynomials-adding-subtracting.html

You can add several polynomials together like that.

Example: Add $2x^2 + 6x + 3xy$, $3x^2 - 5xy - x$, and $6xy + 5$.

If you do not remember how to subtract polynomials, you can also see an animation here:

http://www.mathsisfun.com/algebra/polynomials-adding-subtracting.html

Copy the example here: ____

Example: Calculate $P(x) - Q(x)$, where
$P(x) = x^4 - 6x^3 + 7x - 8$ and
$Q(x) = 2x^3 - 3x^2 + 5x - 1$

3.2. MULTIPLYING POLYNOMIALS.

To multiply a polynomial:

1st Multiply each term in one polynomial by each term in the other polynomial.

2nd Add all the results together, and simplify if needed.

Example: Multiply $3x^2$ and $2x$:

$3 \cdot 2 \cdot x^2 \cdot x = 6x^3$

Example: Multiply the polynomials P(x) and Q(x), where:

$P(x) = 2x^3 - 3x^2 + 5$
$Q(x) = x^2 - 4x + 6$

Other examples:

http://www.mathsisfun.com/algebra/polynomials-multiplying.html

4. REMARKABLE EXPRESSIONS. (= Igualdades notables)

4.1. ADDITION SQUARED. (= Cuadrado de una suma)

First, think about this: Is $(3 + 4)^2$ equal to $3^2 + 4^2$?

So, $\boxed{(a + b)^2 \neq a^2 + b^2}$.

It is very important to learn this law:

The square of an addition is equal to the first term squared, plus two times the first term times the second term, plus the second term squared, *(In Spanish: el cuadrado de una suma es igual al cuadrado del primer sumando, más el doble del primero por el segundo, más el cuadrado del segundo)*:

$$(a + b)^2 = a^2 + 2ab + b^2$$

Example: $(x + 5)^2 = x^2 + 25 + 10x$

It is very easy to prove it. Try it!

4.2. SUBTRACTION SQUARED. (= Cuadrado de una diferencia)

First, think about this: Is $(5 – 3)^2$ equal to $5^2 – 3^2$?

So, $\boxed{(a - b)^2 \neq a^2 - b^2}$

Learn this law:

The square of a subtraction is equal to the first term squared, <u>minus</u> two times the first term times the second term, plus the second term squared *(In Spanish: el cuadrado de una suma es igual al cuadrado del primer sumando, **menos** el doble del primero por el segundo, más el cuadrado del segundo)*:

$$(a + b)^2 = a^2 – 2ab + b^2$$

Example: $(x – 3)^2 = x^2 –6x + 9$. Check it!

4.3. ADDITION TIMES SUBTRACTION. (= Suma por diferencia)

A sum multiplied by a subtraction is equal to the square of the first term **minus** the square of the second term (In Spanish: una suma por una diferencia es igual al cuadrado del primero **menos** el cuadrado del segundo:

$$\mathbf{(a + b)\,(a - b) = a^2 - b^2}$$

Example: $(x + 7)\,(x - 7) = x^2 - 49$. Check it!

4.4. NEWTON'S BINOMIAL.

Here is the formula of Newton's Binomial. Please, do not be frightened; if you read it slowly you will understand more than you think:

$$(a+b)^n = \binom{n}{0} a^n + \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 + \binom{n}{3} a^{n-3} b^3 + \dots + \binom{n}{n} b^n$$

$$(a-b)^n = \binom{n}{0} a^n - \binom{n}{1} a^{n-1} b + \binom{n}{2} a^{n-2} b^2 - \binom{n}{3} a^{n-3} b^3 + \dots + (-1)^n \binom{n}{n} b^n$$

Let's see these examples:

Example 1: Develop the following expression with a power of polynomial:

$$(x+2)^4 = \binom{4}{0} x^4 + \binom{4}{1} x^3 \cdot 2 + \binom{4}{2} x^2 \cdot 2^2 + \binom{4}{3} x \cdot 2^3 + \binom{4}{4} 2^4 =$$

$$= x^4 + 4x^3 \cdot 2 + 6x^2 \cdot 2^2 + 4x \cdot 2^3 + 2^4 = \mathbf{x^4 + 8x^3 + 24x^2 + 32x + 16}$$

Is it clearer now? You do not think so, do you? Try it on your own:

Example 2:

$(x + 1)^3$

Example 3:

$(x - 2)^4$

Example 4:

$(x + y)^5$

Example 5:

$\left(\frac{x}{2} - y\right)^6$

Example 6:

$(2x - y)^{10}$

Example 7:

$\left(x + \frac{1}{x}\right)^{12}$

PRACTISE MORE EXERCISES

EXERCISE 1. Calculate the following operations with monomials:

a) $(3x^4)^3$

b) $-5x^3 + 2x^3 + 4x^3$

c) $-12x^2 : (-4x)$

d) $-6x^2 \cdot (-9x) \cdot x^3$

EXERCISE 2. Calculate the following operations with monomials:

a) $56x^5 : 8x$

b) $6x^3 \cdot (-9x^2)$

c) $-3x^2 + 15x^2 + 4x^2$

d) $(2x^5)^2$

EXERCISE 3. Calculate:

a) $6x^4 \cdot (-9x^3)$

b) $(-3x^3)^3$

c) $5x - 9x + 7x - x$

d) $6x^5 : 4x$

EXERCISE 4. Calculate the following multiplications with polynomials:

a) $(x^5 - 7x^3 + 6x - 1) \cdot 8x^2$

b) $(2x^4 - 8x^2 + 7x - 9) \cdot 7x^3$

c) $(6x^4 + 5x^3 - 8x + 7) \cdot (-9x)$

d) $(x^4 - 9x^3 + 7x - 6) \cdot (-6x^4)$

EXERCISE 5. Simplify the following algebraic expressions:

a) $8x - 12x^2 + 1 + 7x^2 - 3x - 5$

b) $x^2 - 6x - 5x^2 + 7x^2 - 5x - 9$

c) $-7x - 8 + 9x - 11x^2 + 6 + 8x^2$

d) $7x^2 - 9x + 6 - 7x - 8x^2 + 12$

EXERCISE 6. Expand the brackets and simplify the expressions:

a) $7x - (8x^2 + 9 + 5x^2) - 7x - 2$
b) $2x^2 - 5x - 3(2x^2 + 4x^2 - 5x - 6)$
c) $-(3x - 5 + 9x - 7x^2 + 4) + 10x^2$
d) $7(x^2 - 6x + 9) - 7(3x - 7x^2 + 9)$

EXERCISE 7. Extract common factor:

a) $6x - 8y$ b) $8x^3 - 12x^2$
c) $4x^4 + 10x^3 - 6x^2$ d) $9x^2y + 6xy^2 - 3xy$

EXERCISE 8. Here are two polynomials:
$P(x) = 7x^3 - 4x - 1$ $Q(x) = -2x^2 + 5x - 3$

a) Add them:

b) Subtract them:

c) Multiply them:

d) What is the degree of the addition P(x) + Q(x) ?

e) What is the degree of the subtraction P(x) – Q(x)?

f) What is the degree of the multiplication P(x)* Q(x)?

EXERCISE 9. Calculate in your mind using the remarkable expressions:

a) $(x+3)^2$ b) $(x-3)^2$ c) $(x+\sqrt{3})(x-\sqrt{3})$

EXERCISE 10. Calculate in your mind using the remarkable expressions:

a) $\left(2x+\frac{1}{2}\right)^2$

b) $\left(2x-\frac{1}{2}\right)^2$

c) $\left(2x+\frac{1}{2}\right)\left(2x-\frac{1}{2}\right)$

EXERCISE 11. Replace suspension points with the sign = or ≠:

a) $(x-3)^2 \ldots x^2-6x+9$

b) $(x+2)^2 \ldots x^2+4$

c) $(x-3)^2 \ldots x^2-9$

d) $(x+2)^2 \ldots x^2+4x+4$

EXERCISE 12. Extract the common factor of the following expressions:

a) $8x^2-12x$

b) $8x^4+6x^2$

c) $2x^4+4x^3-6x^2$

d) $6x^2y+4xy^2-8xy$

6.DIVISION OF POLYNOMIALS.

To explain how to divide polynomials, we will do some examples, but before that, don´t forget the process for dividing numbers. They are very similar.

Solved example 1:

Divide $D(x) = 6x^5 - 30x^3 + 22x^2 + 27x - 11$ into $d(x) = 2x^3 - 4x^2 + 6$

$$
\begin{array}{rrrrrr|l}
\cancel{6x^5} & & -30x^3 & +22x^2 & +27x & -11 & 2x^3-4x^2+6 \\
\cline{7-7}
-\cancel{6x^5} & +12x^4 & & -18x^2 & & & 3x^2+6x-3 \quad \boxed{\text{Quotient}} \\
\cline{1-4}
 & \cancel{12x^4} & -30x^3 & +4x^2 & +27x & & \\
 & -\cancel{12x^4} & +24x^3 & & -36x & & \\
\cline{2-5}
 & & -\cancel{6x^3} & +4x^2 & -9x & -11 & \\
 & & \cancel{6x^3} & -12x^2 & & +18 & \\
\cline{3-6}
 & & & -8x^2 & -9x & +7 & \boxed{\text{Remainder}}
\end{array}
$$

Just as a matter of interest, an English person makes a division writing it this way:

$$x-2\,\overline{\smash{)}\,x^3+x^2-5x-2}$$

when they are dividing $x^3 + x^2 - 5x - 2$ by $x - 2$.

Example 2: Divide $P(x) = 2x^5 - 8x^4 + 12x^2 + 18$ into $Q(x) = x^2 - 3x - 1$.

Do you remember how to check if a division is correct?

Exercise 13: Calculate a polynomial such that when it is divided into $2x^3 - 5x + 1$ the quotient is $x^2 + 3x - 4$ and the remainder is $-7x^2 + x + 8$.

6.1. RUFFINI'S RULE.

Ruffini's rule has many practical applications; most of them rely on simple division (as demonstrated below) or the common extensions given further below.

If the divisor in a Polynomial division is (x – r), where r is a number, then Ruffini's rule is very useful.

A **worked example** is described below:

$P(x) = 2x^3 + 3x^2 - 4$
$Q(x) = x + 1$

We want to divide P(x) by Q(x) using Ruffini's rule. The main problem is that Q(x) is not a binomial of the form x – r, but rather x + r. We must rewrite Q(x) in this way:

$Q(x) = x + 1$
$Q(x) = x - (-1)$

Now $r = -1$, and we can apply the algorithm:

1st step: Write down the coefficients and n. Note that, as P(x) did not contain a coefficient for x, we have written 0:

```
    |     2     3     0     -4
    |
 -1 |
----|----------------------------
    |
    |
```

2nd step: Move the first coefficient down:

```
    |     2     3     0     -4
    |
 -1 |
----|----------------------------
    |     2
    |
```

3rd step: Multiply the last value obtained by r:

```
    |     2     3     0     -4
    |
 -1 |          -2
----|----------------------------
    |     2
    |
```

4th step: Add the values:

```
    |     2     3     0     -4
    |
 -1 |          -2
----|----------------------------
    |     2     1
    |
```

5 th step: Repeat steps 3 and 4 until we've finished:

```
    |     2     3     0     -4
    |
 -1 |          -2    -1      1
----|----------------------------
    |     2     1    -1     -3
    |
     {result coefficients}{remainder}
```

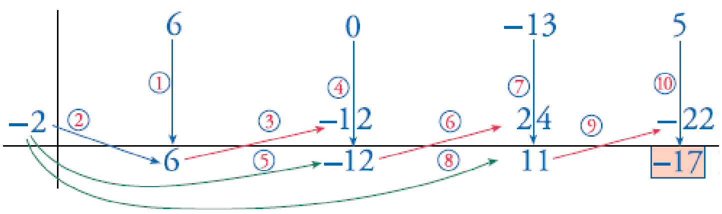
Remainder

Quotient

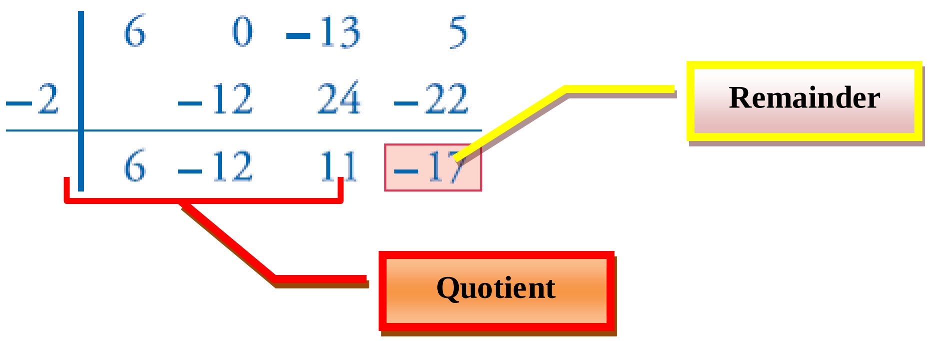
So, if the original number = divisor × quotient + remainder, then

$$P(x) = Q(x) \cdot R(x) + s, \text{ where}$$
$$R(x) = 2x^2 + x - 1 \text{ and } s = -3$$

Solved example 1: Divide $6x^3 - 13x + 5$ into $x + 2$



Therefore:



Quotient: $C(x) = 6x^2 - 12x + 11$.

Remainder: $R = -17$.

Example 3: Divide $P(x) = 2x^3 - 13x + 8$ into $Q(x) = x + 3$ using Ruffini´s rule.

7. THE REMAINDER THEOREM.

The remainder in a division like $P(x) : (x - a)$, where a is any number, is the numeric value of the polynomial for $x = a$, it is $P(a)$.

Solved example 1: Without doing the division, what is the remainder if we divide $P(x) = x^3 - 7x + 15$ by $x + 3$?

$$R = P(-3) = (-3)^3 - 7 \cdot (-3) + 15 = -27 + 21 + 15 = 36 - 27 = 9$$

Example 2: Without doing the division, work out the remainder of the division $P(x) = x^3 - 6x^2 + 5$ by $x - 2$.

Example 3: Without doing the division, what is the value of the remainder of the division $P(x) = x^4 + 3x^3 - 5x - 7$ by $x + 3$?

Example 4: Calculate the value of k, if the remainder of the division $(x^3 + kx - 6) : (x - 2)$ is 5.

8. POLYNOMIAL ROOTS. (Raíces de polinomios)

A **root** of a polynomial $P(x)$ is a number a such that $P(a)=0$.

The fundamental theorem of algebra states that a polynomial $P(x)$ of degree n has exactly n roots, some of which may be degenerate. For example, the roots of the polynomial

$$x^3 - 2x^2 - x + 2 = (x-2)(x-1)(x+1)$$

are 2, 1, and −1. Finding the roots of a polynomial is therefore equivalent to polynomial factorisation into factors of degree 1.

Example 1: Is number 3 a root of the polynomial $P(x)= x^3 + x^2 - 9x - 9$? And number -3?

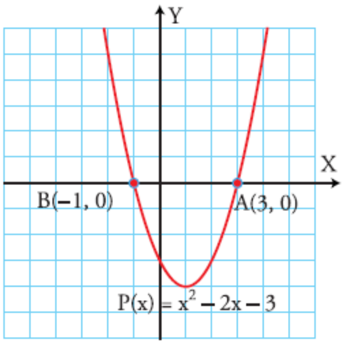
Example 2: Calculate the roots of polynomial $P(x)= 2x^2 - 8x + 6$. Also, do the polynomial factorisation.

Example 3: Calculate the roots of polynomial $P(x)= x^2 + 9$. Also, do the polynomial factorisation.

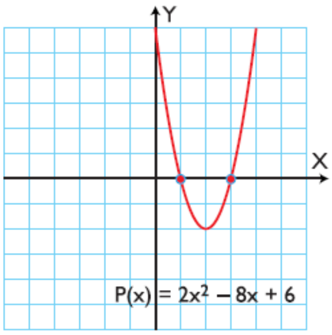
Example 4. Do the polynomial factorisation of $P(x)= x^3 + 2x^2 - 5x - 6$.

Example 5: This example shows the graphic interpretation of roots of polynomials.

The roots of a polynomial are the points of intersection of its graphical representation with the X axis.



Example 6: Point out the roots of the polynomial below:



PRACTISE MORE EXERCISES

Exercise 14. Divide $P(x)= 6x^6 – 13x^5 – 20^3 + 50x^2 – 4$ **into** $Q(x)= 2x^3 – 3x^2 + 1$.

Exercise 15. Divide $P(x)= x^4 – 6x^2 + 4x + 5$ **into** $Q(x)= x + 2$ **using Ruffini's rule.**

Exercise 16. Without doing the division, calculate the remainder of the division $P(x)= x^3 – 5x^2 + 7$ **by** $Q(x)= x – 3$.

Exercise 17. Without doing the division, calculate the remainder of the division
$P(x)= x^4 – 2x^3 + 7x – 3$ **by** $Q(x)= x + 2$.

Exercise 18. Is number 2 a root of polynomial $P(x)= x^3 + 2x^2 – x – 2$**? And number** $–2$**?**

Exercise 19. Is the polynomial $P(x)= x^4 – 6x^3 + 8x^2 – 6x – 9$ **divisible by** $Q(x) = x – 3$**?**

Exercise 20. Calculate the value of k in order that the remainder is 7, in the division $(x^4 + kx^2 - 5x + 6) : (x + 1)$.

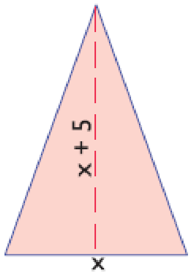
Exercise 21. Do the polynomial factorisation of the following polynomials:

a) $24x^3 - 18x^2$ b) $2x^3 + 12x^2 + 18x$

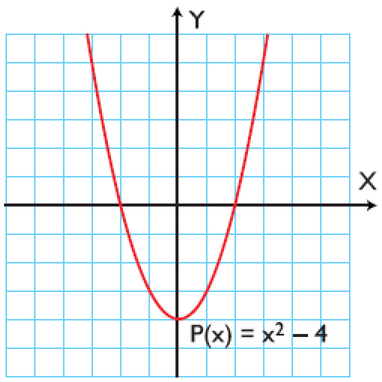
c) $9x^2 - 4$ d) $5x^4 - 10x + 5x^2$

Exercise 22. Calculate the value of k in order that $P(x) = x^3 + 5x^2 + kx - 8$ **is divisible by** $Q(x) = x + 2$**.**

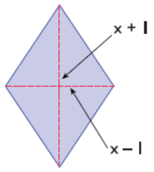
Exercise 23. Calculate the polynomial which is the area of this triangle:



Exercise 24. Look at the graphical representation of the polynomial $P(x) = x^2 - 4$ **and point out its roots.**



Exercise 25. Calculate the polynomial which is the area of the rhombus below:



9. ALGEBRAIC FRACTIONS

An algebraic fraction is a division of two polynomials:

$$\frac{P(x)}{Q(x)} \qquad Q(x) \neq 0$$

For example:

$$\frac{x^3 + 2x + 4}{x^2 - 1} \qquad \frac{7x + 9}{x^3 - 5x} \qquad \frac{3}{x - 2}$$

9.1. SIMPLIFYING ALGEBRAIC FRACTIONS

To simplify an algebraic fraction we have to factorise its denominator and numerator, and eliminate their common factors.

Example: Simplify the algebraic fraction $\frac{x^3 + 6x^2 + 9x}{x^2 - 9}$:

$$\frac{x^3 + 6x^2 + 9x}{x^2 - 9} = \frac{x(x^2 + 6x + 9)}{(x + 3)(x - 3)} = \frac{x(x + 3)^{\cancel{2}}}{\cancel{(x + 3)}(x - 3)} = \frac{x(x + 3)}{(x - 3)}$$

9.2. ADDITION AND SUBTRACTION WITH ALGEBRAIC FRACTIONS

To add or subtract algebraic fractions we follow the same rules as for fractions of numbers.
Study the following example (notice that to get the same denominator we calculate the l.c.m. of the denominators):

$$\frac{x}{x - 1} + \frac{4x - 5}{x^2 - x} - \frac{x + 2}{x} = \frac{x^2 + 4x - 5 - (x - 1)(x + 2)}{x(x - 1)} =$$

$$\boxed{\text{m.c.m.}(x - 1, x^2 - x, x) = x(x - 1)}$$

$$= \frac{\cancel{x^2} + 4x - 5 - \cancel{x^2} - x + 2}{x(x - 1)} = \frac{3x - 3}{x(x - 1)} = \frac{3\cancel{(x - 1)}}{x\cancel{(x - 1)}} = \frac{3}{x}$$

9.3. MULTIPLICATIONS AND DIVISIONS WITH ALGEBRAIC FRACTIONS.

To multiply and divide algebraic fractions we use the same rules as for fractions of numbers.

Example: Multiply these algebraic fractions: $\frac{x + 3}{x - 1} \cdot \frac{x^2 + 2x + 1}{x + 2}$

$$\frac{x + 3}{x - 1} \cdot \frac{x^2 + 2x + 1}{x + 2} = \frac{(x + 3)(x^2 + 2x + 1)}{(x - 1)(x + 2)} = \frac{x^3 + 5x^2 + 7x + 3}{x^2 + x - 2}$$

Example: Divide these algebraic fractions: $\frac{x^2-4}{x-1} : \frac{x+2}{x} =$

$$\frac{x^2-4}{x-1} : \frac{x+2}{x} = \frac{\cancel{(x+2)}(x-2)}{x-1} \cdot \frac{x}{\cancel{x+2}} = \frac{x^2-2x}{x-1}$$

EXERCISES:

Exercise 26. Factorise the numerator and denominator mentally and simplify the following algebraic fractions:

a) $\frac{x^2-x}{3x-3}$

b) $\frac{x^2-4x+4}{x^2-4}$

c) $\frac{(x+2)^2}{x^2-4}$

d) $\frac{x^2}{x^2-x}$

e) $\frac{4x^2-9}{2x-3}$

f) $\frac{9x^2+6x+1}{3x+1}$

Exercise 27. Complete the following equalities:

a) $\frac{x+1}{x^2-2x-3} = \frac{2x-4}{\cdots}$

b) $\frac{x^2-x-2}{x^2-6x+8} = \frac{\cdots}{x-4}$

Exercise 28. Calculate:

a) $\dfrac{2}{x} + \dfrac{1}{x+1}$

b) $\dfrac{3}{x^2-4} - \dfrac{x}{x-2}$

c) $\dfrac{2}{x+3} + \dfrac{2}{x-3}$

d) $\dfrac{8}{x^2+2x} - \dfrac{4x}{2x+4}$

e) $\dfrac{1}{x^2} - \dfrac{x+1}{x^2+x}$

f) $\dfrac{1}{2x-1} - \dfrac{x+1}{(2x-1)^2}$

Exercise 29. Calculate:

a) $\dfrac{x+1}{x-2} \cdot \dfrac{x^2}{x^2-1}$

b) $\dfrac{x+2}{x+1} \cdot \dfrac{x^2+x}{x^2-4}$

c) $\frac{2x}{x-2} \cdot \frac{x^2-4}{2}$

d) $\frac{3x+3}{3x} \cdot \frac{x^2-3}{x^2-9}$

Exercise 30. Calculate:

a) $\frac{x+3}{x+2} : \frac{x^2-9}{x^2-4}$

b) $\frac{2x^2+x}{x^2-1} : \frac{2x+1}{3x^2-4}$

c) $\frac{3x}{2x-2} : \frac{2x}{x-1}$

d) $\frac{x^2-x}{x-3} : \frac{4x-4}{x^2-9}$

Exercise 31. Calculate and simplify:

a) $\left(x + \frac{4x-1}{x-4}\right)\frac{2}{x-1}$

b) $\left(\frac{1}{x^2-4} + \frac{1}{x-2}\right) : \left(1 + \frac{2}{x-2}\right)$

c) $\left(\frac{1}{x} + \frac{2}{x^2}\right)\frac{3x^2}{x+2}$

d) $\left(x + \frac{x}{1-x}\right) : \left(x - \frac{x}{1-x}\right)$