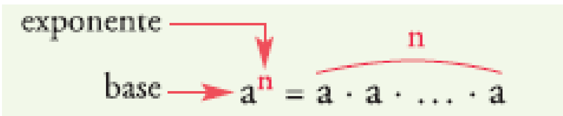
UNIT 2. POWERS, ROOTS AND LOGARITHMS.

1. POWERS.

1.1. DEFINITION.

When you multiply two or more numbers, each number is called a **factor of the product**. When the same factor is repeated, you can use an **exponent** to simplify your writing. An exponent tells you how many times a number, called the **base**, is used as a factor.
A power is a number that is expressed using exponents.



In English: base → ……………………………… Exponente →…………………………

Other examples:

- **5^2 = *5 al cuadrado* = five to the second power or five squared**
- **5^3 = *5 al cubo* = five to the third power or five cubed**
- **4^5 = *4 elevado a la quinta potencia* = four (raised) to the fifth power**
- **15^{21} = fifteen to the twenty-first**
- **33^{22} = thirty-three to the power of twenty-two**

Exercise 1. Calculate:

a) $(-2)^3 =$

f) $2^3 =$

b) $(-3)^3 =$

g) $(-1)^4 =$

c) $(-5)^4 =$

h) $(-5)^3 =$

d) $(-10)^3 =$

i) $(-10)^6 =$

e) $(7)^3 =$

j) $(-7)^3 =$

Exercise: Calculate with the calculator:

a) $(-6)^2 =$ b) $5^3 =$ c) $(2)^{20} =$
d) $(10)^8 =$ e) $(-6)^{12} =$

For more information, you can visit http://en.wikibooks.org/wiki/Basic_Algebra

1.2. PROPERTIES OF POWERS.

Here are the properties of powers. Pay attention to the last one (section vii, powers with negative exponent) because it is something new for you:

i) **Multiplication of powers with the same base:** $a^n \cdot a^p = a^{n+p}$

E.g.: $7^2 \cdot 7^3 = 7^5$ | $7^2 \cdot 7^3 = \overbrace{7 \cdot 7}^{2} \cdot \overbrace{7 \cdot 7 \cdot 7}^{3} = 7^5$

ii) **Division of powers with the same base :** $\dfrac{a^n}{a^p} = a^{n-p}$

E.g.: $\dfrac{6^5}{6^3} = 6^2$ | $\dfrac{6^5}{6^3} = \dfrac{\cancel{6} \cdot \cancel{6} \cdot \cancel{6} \cdot 6 \cdot 6}{\cancel{6} \cdot \cancel{6} \cdot \cancel{6}} = 6 \cdot 6 = 6^2$

E.g.: $3^5 : 3^4 = 3^1 = 3$

iii) **Power of a power:** $(a^n)^p = a^{n \cdot p}$

E.g. $(3^5)^2 = 3^{10}$

Checking: $(3^5)^2 = 3^5 \cdot 3^5 = (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) \cdot (3 \cdot 3 \cdot 3 \cdot 3 \cdot 3) = 3^{10}$

iv) **Power of a multiplication:** $(a \cdot b)^n = a^n \cdot b^n$

E.g. $(3 \cdot 5)^3 = 3^3 \cdot 5^3$

v) **Power of a division :** $(a : b)^n = a^n : b^n$

E.g.: $(3 : 5)^3 = 3^3 : 5^3 = 27 : 125$

vi) **Remember this:** $a^0 = 1$, **so any number powered 0 is equal to 1.**

Examples: $5^0 = 1$, $2^0 = 1$, $(0.5)^0 = 1$, $(-5)^0 = 1$

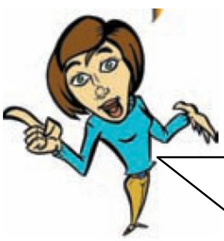
vii) **Powers with a negative exponent**. $x^{-n} = \dfrac{1}{x^n}$

Example 1: 3^{-3} = ***tres a la menos tres*** = **three to the negative third power = one over three cubed**

Example 2: $4^{-2} = \dfrac{1}{4^2} = \dfrac{1}{16}$ **, here is why:**

$$4^{-2} = 4^{0-2} = \frac{4^0}{4^2} = \frac{1}{4^2}$$

$$\left\{ \begin{array}{l} \dfrac{4^3}{4^5} = 4^{3-5} = 4^{-2} \\ \dfrac{4^3}{4^5} = \dfrac{\cancel{4} \cdot \cancel{4} \cdot \cancel{4}}{\cancel{4} \cdot \cancel{4} \cdot \cancel{4} \cdot 4 \cdot 4} = \dfrac{1}{4^2} \end{array} \right.$$



To revise these rules, you can visit this video on the Internet:

http://www.math-videos-online.com/exponents-rules.html

Exercise 1: The most common errors with powers are in the following examples, find them:

a) $2^3 = 6$?

b) $3^0 = 0$?

c) $-2^2 = -4$?

d) $(2+3)^2 = 2^2 + 3^2$?

e) $(3-1)^2 = 3^2 - 1^2$?

f) $(3)^{-2} = -3^2$?

Exercise 2: Calculate in your mind:

a) $(3)^0 =$ b) $(-3)^1 =$ c) $(-3)^2 =$ d) $(-3)^3 =$ e) $(-3)^4 =$

Exercise 3: Calculate in your mind:

a) $-2^3 =$ b) $-3^3 =$ c) $-2^4 =$ d) $-3^4 =$ e) $-10^2 =$

Exercise 4: Use the properties of powers to calculate:

a) $5^3 \cdot 5^4 =$ b) $5^9 : 5^3 =$ c) $(5^3)^2 =$

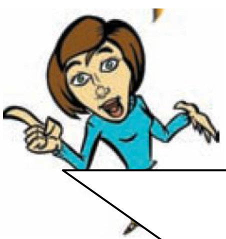
d) $5^3 \cdot 7^3 =$ e) $5^4 : 7^4 =$

Exercise 5: Write as a power with an integer base:

a) $\frac{1}{5^3}$ b) $\frac{1}{16}$ c) $\frac{1}{3^2}$ d) $\frac{1}{81}$

Exercise 6. Write as a power:

a) $x^3 \cdot x^4 =$ b) $x^7 : x^3 =$ c) $(x^3)^2 =$ d) $x^3 \cdot x^4 : x^5 =$



To practise with exponents, you can visit this website:

http://www.mathsisfun.com/algebra/negative-exponents.html

2. ROOTS.

2.1. SQUARE ROOT.

First, do not forget:

We usually write

$\sqrt{4}=2$ $\sqrt{1}=1$ $\sqrt{9}=3$

But this is not absolutely true, look at this carefully: $\sqrt{a}=b$ if $b^2=a$

and so:

$\sqrt{4}=\pm 2$ because $2^2=4$ and $(-2)^2=4$

$\sqrt{9}=\pm 3$ because $(3)^2=9$ and $(-3)^2=9$

$\sqrt{1}=\pm 1$ because $(1)^2=1$ and $(-1)^2=1$

$\sqrt{0}=0$

$\sqrt{-9}=\nexists$ (it does not exist)

So, **a number can have two square roots, one, or none.**

E.g.: How many roots has $\sqrt{4}$ got ? Two roots, they are 2 and -2, because $2^2=4$ and $(-2)^2=4$

E.g.: How many roots has $\sqrt{-16}$ got?

E.g.: How many roots has $\sqrt{0}$ got ?

E.g.: How many roots has $\sqrt{81}$ got?

LET'S APPROXIMATE SQUARE ROOTS:

a) $5<\sqrt{27}<6$ because $5^2=25$ *and* $6^2=36$

b) .... $<\sqrt{10}<$ .... because

c) .... $<\sqrt{17}<$ .... because

d) .... $<\sqrt{70}<$ .... because

PROPERTIES OF SQUARE ROOTS.

i) $\sqrt{a}\cdot\sqrt{b}=\sqrt{a\cdot b}$. Example: $\sqrt{12}\cdot\sqrt{3}=\sqrt{12\cdot 3}=\sqrt{36}=\pm 6$

ii) $\dfrac{\sqrt{a}}{\sqrt{b}}=\sqrt{\dfrac{a}{b}}$. Example: $\dfrac{\sqrt{12}}{\sqrt{3}}=\sqrt{\dfrac{12}{3}}=\sqrt{4}=\pm 2$

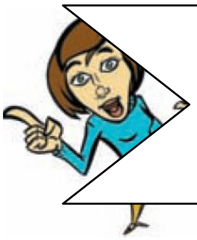
N.B.: *COMMON MISTAKES!*

- $\sqrt{a+b}\neq\sqrt{a}+\sqrt{b}$. Example: $\sqrt{9+16}\neq\sqrt{9}+\sqrt{16}$ because $\sqrt{25}=5\neq 3+4$.
- $\sqrt{a-b}\neq\sqrt{a}-\sqrt{b}$. Example: $\sqrt{25-9}\neq\sqrt{25}-\sqrt{9}$ because $\sqrt{16}=4\neq 5-3$.

EXTRACTING THE FACTORS OF A ROOT:

Examples:

- $\sqrt{12}=\sqrt{4\cdot 3}=\sqrt{2^2\cdot 3}=2\sqrt{3}$
- $\sqrt{50}=\sqrt{25\cdot 2}=\sqrt{5^2\cdot 2}=5\sqrt{2}$
- $\sqrt{18}=\sqrt{9\cdot 2}=\sqrt{3^2\cdot 2}=3\sqrt{2}$
- $\sqrt{62}=$
- $\sqrt{75}=$
- $\sqrt{200}=$
- $\sqrt{20}=$
- $\sqrt{45}=$
- $\sqrt{48}=$
- $\sqrt{2^2\cdot 3^2\cdot 5}=$
- $\sqrt{7^2\cdot 3^4}=$
- $\sqrt{2^6\cdot 5^4}=$



To revise how to simplify roots, you can visit this video on the Internet:

http://www.math-videos-online.com/simplifying-square-roots.html

2.2. CUBE ROOT.

- $\sqrt[3]{8}=2$ because $2^3=8$
- $\sqrt[3]{27}=3$ because $3^3=27$
- $\sqrt[3]{1}=1$ because $1^3=1$
- $\sqrt[3]{125}=$ because
- $\sqrt[3]{0}=$ because
- $\sqrt[3]{-8}=$ because

LET US APPROXIMATE CUBE ROOTS:

a) $1<\sqrt[3]{4}<2$ because $1^3=1$ and $2^3=8$.

b) .... $<\sqrt[3]{8}<$ .... because

c) .... $<\sqrt[3]{33}<$ .... because

d) .... $<\sqrt[3]{77}<$ .... because

PROPERTIES OF CUBE ROOTS.

i) $\sqrt[3]{a}\cdot\sqrt[3]{b}=\sqrt[3]{a\cdot b}$. Example: $\sqrt[3]{25}\cdot\sqrt[3]{5}=\sqrt[3]{125}=5$

ii) $\dfrac{\sqrt[3]{a}}{\sqrt[3]{b}}=\sqrt[3]{\dfrac{a}{b}}$. Example: $\dfrac{\sqrt[3]{24}}{\sqrt[3]{3}}=\sqrt[3]{\dfrac{24}{3}}=\sqrt[3]{8}=2$

2.3. UMPTEENTH ROOT (RAÍZ n-ésima)

The umpteenth root of a number "a" is another number "b" so that $b^n = a$.
So:

$\sqrt[n]{a} = b$ if $b^n = a$

Example:

$\sqrt[4]{81}=\pm 3=\begin{cases}3\\-3\end{cases}$ because $\begin{cases}3^4=81\\(-3)^4=81\end{cases}$

2.4. EQUIVALENT ROOTS

Two roots are equivalent if they have got the same solutions. To get equivalent roots you can multiple or divide the index and exponent by the same number.

Example:

$$\sqrt[3]{5^2}=\sqrt[6]{5^4}=\sqrt[9]{5^6}=\sqrt[12]{5^8}=\ldots=2{,}92\ldots$$

Example: Simplifying a root.

$$\sqrt[18]{5^{12}}=\sqrt[18:6]{5^{12:6}}=\sqrt[3]{5^2}$$

M.C.D. (12, 18) = 6

2.5. PUTTING FACTORS IN A ROOT.

To put a number into a root you have to raise it to the power of the index of the root.

Example:

$$2\sqrt[3]{5}=\sqrt[3]{2^3\cdot 5}=\sqrt[3]{8\cdot 5}=\sqrt[3]{40}$$

Other examples:

2.6. EXTRACTING FACTORS OF A ROOT:

Let's learn how to extract the factors of a root using some examples:

Example 1:

$$\sqrt[3]{32a^{17}b^{12}c^7}=\sqrt[3]{2^5a^{17}b^{12}c^7}=2a^5b^4c^2\sqrt[3]{2^2a^2c}$$

Example 2:

$\sqrt[3]{40} = \sqrt[3]{8 \cdot 5} = \sqrt[3]{2^3 \cdot 5} = 2\sqrt[3]{5}$

Example 3:

$\sqrt[3]{250} = \sqrt[3]{125 \cdot 2} = \sqrt[3]{5^3 \cdot 2} = 5\sqrt[3]{2}$

Other examples:

- $\sqrt[3]{500} =$
- $\sqrt[3]{54} =$
- $\sqrt[3]{40} =$
- $\sqrt[3]{1000} =$
- $\sqrt[3]{135} =$
- $\sqrt{81a^5bc^6} =$
- $\sqrt[3]{128a^8b^2c^{15}} =$
- $\sqrt{243a^8b^3c^7} =$
- $\sqrt[3]{125a^9b^{17}c^{25}} =$

2.7. ADDITION AND SUBTRACTION OF ROOTS

Two roots are similar (in Spanish: *radicales semejantes*) if they have got the same index and the same radicant. We **cannot add two roots if they are not similar roots**.

Let's study the following examples:

Example: $4\sqrt{50} - 7\sqrt{8} + 5\sqrt{18} =$

Example: $\sqrt{50} - \sqrt{32} + \sqrt{18}$

Example: $5\sqrt{98} - 3\sqrt{200} + 4\sqrt{8}$

2.8. PROPERTIES OF ROOTS

FORMULA	EXAMPLE
$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$	$\sqrt[3]{5} \cdot \sqrt[3]{25} = \sqrt[3]{5 \cdot 25} = \sqrt[3]{125} = 5$
$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}$	$\sqrt[3]{32} : \sqrt[3]{4} = \sqrt[3]{32 : 4} = \sqrt[3]{8} = 2$
$(\sqrt[n]{a})^p = \sqrt[n]{a^p}$	$(\sqrt[3]{5})^2 = \sqrt[3]{5^2}$
$\sqrt[n]{\sqrt[p]{a}} = \sqrt[n \cdot p]{a}$	$\sqrt[3]{\sqrt{5}} = \sqrt[6]{5}$

2.9. SUMMARY OF THE PROPERTIES OF POWERS AND ROOTS.

POWERS		ROOTS	
$a^n = a \cdot a \cdot \ldots^{n)} \ldots \cdot a$	$2^3 = 2 \cdot 2 \cdot 2 = 8$	$\sqrt{a} = b$ si $b^2 = a$	$\sqrt{25} = \pm 5$
$0^n = 0,\ 0 \neq 0$	$0^5 = 0$	$\sqrt[3]{a} = b$ si $b^3 = a$	$\sqrt[3]{8} = 2$
$1^n = 1$	$1^3 = 1$	$\sqrt[n]{a} = b$ si $b^n = a$	$\sqrt[5]{32} = 2$
$a^0 = 1,\ a \neq 0$	$5^0 = 1$	$\sqrt[n]{a^n} = \left(\sqrt[n]{a}\right)^n = a$	$\sqrt[5]{7^5} = \left(\sqrt[5]{7}\right)^5 = 7$
$a^1 = a$	$4^1 = 4$	$\sqrt[ns]{a^{ps}} = \sqrt[n]{a^p}$	$\sqrt[10]{7^6} = \sqrt[5 \cdot 2]{7^{3 \cdot 2}} = \sqrt[5]{7^3}$
$a^n \cdot a^p = a^{n+p}$	$5^3 \cdot 5^4 = 5^7$	$a\sqrt[n]{b} = \sqrt[n]{a^n \cdot b}$	$2\sqrt[3]{5} = \sqrt[3]{2^3 \cdot 5} = \sqrt[3]{40}$
$a^n : a^p = a^{n-p}$	$2^8 : 2^3 = 2^5$	$\sqrt[n]{a} \cdot \sqrt[n]{b} = \sqrt[n]{a \cdot b}$	$\sqrt[5]{2} \cdot \sqrt[5]{3} = \sqrt[5]{6}$
$(a^n)^p = a^{n \cdot p}$	$(5^3)^2 = 5^6$	$\sqrt[n]{a} : \sqrt[n]{b} = \sqrt[n]{a : b}$	$\sqrt[3]{2} : \sqrt[3]{5} = \sqrt[3]{2 : 5}$
$(a \cdot b)^n = a^n \cdot b^n$	$(2 \cdot 3)^5 = 2^5 \cdot 3^5$	$\left(\sqrt[n]{a}\right)^p = \sqrt[n]{a^p}$	$\left(\sqrt[3]{7}\right)^2 = \sqrt[3]{7^2}$
$(a : b)^n = a^n : b^n$	$(5 : 7)^3 = 5^3 : 7^3$	$\sqrt[n]{\sqrt[p]{a}} = \sqrt[n \cdot p]{a}$	$\sqrt[5]{\sqrt[3]{7}} = \sqrt[15]{7}$
$a^{-n} = \frac{1}{a^n}$	$2^{-3} = \frac{1}{2^3}$	$\sqrt[n]{a} = a^{1/n}$	$\sqrt[3]{7} = 5^{1/3}$
$a^{1/n} = \sqrt[n]{a}$	$6^{1/5} = \sqrt[5]{6}$	$\frac{1}{\sqrt[n]{a}} = a^{-1/n}$	$\frac{1}{\sqrt[5]{2}} = 2^{-1/5}$
$a^{-1/n} = \frac{1}{\sqrt[n]{a}}$	$5^{-1/4} = \frac{1}{\sqrt[4]{5}}$	$\sqrt[n]{a^p} = a^{p/n}$	$\sqrt[3]{5^2} = 5^{2/3}$
$a^{p/n} = \sqrt[n]{a^p}$	$7^{2/3} = \sqrt[3]{7^2}$	$\frac{1}{\sqrt[n]{a^p}} = a^{-p/n}$	$\frac{1}{\sqrt[4]{7^3}} = 7^{-3/4}$
$a^{-p/n} = \frac{1}{\sqrt[n]{a^p}}$	$6^{-3/4} = \frac{1}{\sqrt[4]{6^3}}$		

2.10. FRACTIONAL EXPONENT AND ROOTS.

Do not forget the general **rule**:

$x^{1/2}$ = the **square root** of x = $\sqrt{x}$

$x^{1/4}$ = The 4th Root of x = $\sqrt[4]{x}$

So we can come up with a general rule:

A fractional exponent like 1/n means to take the n-th root:	$x^{\frac{1}{n}} = \sqrt[n]{x}$

What About More Complicated Fractions?

What about a fractional exponent like $4^{3/2}$? That is a way to say: "do a cube (3) and a square root (1/2)", in any order. Here is the explanation:

A fraction (like m/n) can be broken into two parts:

- a whole number part (m) , and
- a fraction (1/n) part

So, because m/n = m · (1/n), we can do this:

$$x^{\frac{m}{n}} = x^{\left(m \times \frac{1}{n}\right)} = \left(x^m\right)^{\frac{1}{n}} = \sqrt[n]{x^m}$$

And we get this:

A fractional exponent like m/n means: Do the m-th power, then take the n-th root OR Take the n-th root and then do the m-th power	$x^{\frac{m}{n}} = \sqrt[n]{x^m}$ $= \left(\sqrt[n]{x}\right)^m$

To put it another way:

$$a^{p/n} = \sqrt[n]{a^p},\ a > 0 \qquad a^{-p/n} = \frac{1}{\sqrt[n]{a^p}},\ a > 0$$

Some examples:

Example 1: What is $4^{3/2}$?

$$\mathbf{4^{3/2}} = 4^{3\times(1/2)} = \sqrt{(4^3)} = \sqrt{(4\times4\times4)} = \sqrt{(64)} = \mathbf{8}$$

or

$$\mathbf{4^{3/2}} = 4^{(1/2)\times3} = (\sqrt{4})^3 = (2)^3 = \mathbf{8}$$

Either way gets the same result.

Exercise 1. Write as a root:

a) $\sqrt[5]{3}$ b) $\dfrac{1}{\sqrt[6]{5}}$ c) $\sqrt[7]{3^5}$ d) $\dfrac{1}{\sqrt[3]{7^2}}$

Exercise 2. Write as a root and calculate the result:

a) $27^{1/3} =$ b) $49^{-1/2} =$ c) $128^{3/7}$ d) $243^{-2/5}$

Exercise 3. Use the properties of roots to write these operations with just one root:

a) $\sqrt{5}\cdot\sqrt{3}$ b) $\sqrt{6}:\sqrt{3}$

c) $(\sqrt[3]{5})^2$ d) $\sqrt[3]{\sqrt{5}}$

Exercise 4. Use the properties of roots to calculate the result:

a) $\sqrt{6}\cdot\sqrt{6}$ b) $\sqrt{20}:\sqrt{5}$

c) $\sqrt[3]{25}\cdot\sqrt[3]{5}$ d) $\sqrt{\sqrt[3]{64}}$

Exercise 5. Calculate:

a) $\sqrt{3}\cdot\sqrt{6}$ b) $\sqrt[3]{12}\cdot\sqrt[3]{10}$

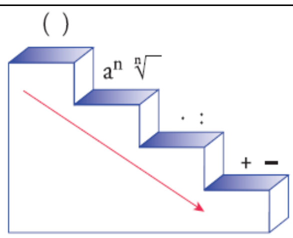
c) $\sqrt{3}\cdot\sqrt[3]{2}$ d) $\sqrt[4]{5}\cdot\sqrt[6]{3}$

Exercise 6. Calculate:

a) $\sqrt{6}:\sqrt{3}$ b) $\sqrt[3]{40}:\sqrt[3]{5}$

c) $\sqrt[3]{9}:\sqrt{12}$ d) $\sqrt[3]{2}:\sqrt[5]{3}$

REMINDER:

	1. Brackets. 2. Powers and roots. 3. Multiplications and divisions. 4. Additions and subtractions.

Example:

$(56 - 7^2)\cdot\sqrt{25} = (56 - 49)\cdot 5 = 7\cdot 5 = 35$

With the calculator: $\boxed{(}\ \boxed{56}\ \boxed{-}\ \boxed{7}\ \boxed{x^2}\ \boxed{)}\ \boxed{\times}\ \boxed{\sqrt{\ }}\ \boxed{25}\ \boxed{=}$ **35**

Exercise 7: Work out in your mind and check it with the calculator:

a) $\left(9^2 + 23 - 7^2\right) \cdot \sqrt{64}$

b) $\left(10^2 - \sqrt{81} + 5^3\right) : \sqrt{36}$

Exercise 8: Indicate if they are equivalent roots:

a) $\sqrt[4]{3^6}$ y $\sqrt{3^3}$

b) $\sqrt[5]{2^{10}}$ y $\sqrt{2}$

c) $\sqrt[4]{36}$ y $\sqrt{6}$

d) $\sqrt[4]{5^{10}}$ y $\sqrt{5^4}$

Exercise 9: Calculate and simplify:

a) $4\sqrt{27} \cdot 5\sqrt{6}$

b) $\left(\dfrac{\sqrt[6]{32}}{\sqrt{8}}\right)^3$

c) $\sqrt[3]{2} \cdot \sqrt{3}$

d) $\dfrac{\sqrt{3} \cdot \sqrt[3]{3}}{\sqrt[4]{3}}$

Exercise 10: Calculate the value of the following roots:

a) $\sqrt[4]{81}$ b) $\sqrt[3]{-27}$ c) $\sqrt[5]{-100.000}$ d) $\sqrt[3]{-216}$ e) $\sqrt[4]{625}$ f) $\sqrt[7]{-128}$

2.11. RATIONALISE.

To rationalise a quantity means, literally, to make it rational. A rational number is one that can be expressed as the ratio of two integers, like 2/3, for example, or 4, since 4 can be expressed as 4/1. The quantity 2.5 is also rational, since it represents 2 and 1/2, or 5/2. In fact, any number with a limited decimal part is rational. Any number whose decimal part begins to repeat is also rational, such as .33333333...., since this can be expressed as 1/3.

Numbers that are not rational are called irrational. Examples of irrational numbers are the square root of 2, pi, and e. The decimal parts of these numbers are infinite and never repeat. For example, $\sqrt{2} = 1.41421356237309504881688724209\ldots$

So, to rationalise the denominator of a fraction, we need to re-write the fraction so that our new fraction has the same value as the original, and it has a rational denominator. The standard method of changing a fraction into an equivalent fraction with an integer as its denominator is to multiply it by some number over itself, since any non-zero number over itself is 1, and multiplication by 1 does not change its value:

In other words, sometimes in algebra it is desirable to find an equivalent expression for a radical expression that does not have any radicals in the denominator. This process is called rationalising the denominator. You can also do the same thing with the numerator, but it is more common to rationalise the denominator. We will only explain how to rationalise the denominator, and if you ever need to rationalise a numerator you can do the same thing with the numerator.

There are 3 cases that commonly come up in algebra lessons:

1. single square root
2. single higher root
3. sums and differences of square roots

Type of Problem	Example	Solution
The denomina-tor is a one-term square root	Rationalize the denominator of : $\frac{5}{\sqrt{2}}$	$\frac{5}{\sqrt{2}} \times \frac{\sqrt{2}}{\sqrt{2}} = \frac{5\sqrt{2}}{\left(\sqrt{2}\right)\left(\sqrt{2}\right)} = \frac{5\sqrt{2}}{2}$

Type of Problem	Example	Solution
The denomina-tor is a one-term root (not a square root)	Rationalize the denominator: $\dfrac{2}{\sqrt[3]{5}}$	Since the index of the radical is 3, this means to come out from under the radical, we need 3 like factors, all under the radical of index 3. We only have one factor of 5, so this means we should multiply by $\sqrt[3]{(5)(5)}$, because $\sqrt[3]{5}\times\sqrt[3]{(5)(5)}=\sqrt[3]{(5)(5)(5)}=5.$ So, we proceed with our solution: $\dfrac{2}{\sqrt[3]{5}}\cdot\dfrac{\sqrt[3]{(5)(5)}}{\sqrt[3]{(5)(5)}}=\dfrac{2\cdot\sqrt[3]{(5)(5)}}{5}=\dfrac{2\sqrt[3]{25}}{5}$
The denomina-tor has two terms (it is a binomial)	Rationalize the denominator: $\dfrac{2}{1+\sqrt{3}}$	Since our denominator is a binomial, one term of which is a radical, our strategy will be to consider the denominator as *one* of the binomial factors of a difference of two squares, and to multiply by the *other* difference-of-two-squares binomial factor. This way the "outer" and "inner" products will be additive opposites, and so combine to make 0. So, we proceed with our solution: $\dfrac{2}{1+\sqrt{3}}\cdot\dfrac{1-\sqrt{3}}{1-\sqrt{3}}=\dfrac{2\left(1-\sqrt{3}\right)}{\left(1+\sqrt{3}\right)\left(1-\sqrt{3}\right)}=\dfrac{2\left(1-\sqrt{3}\right)}{1-3}$ $=\dfrac{2\left(1-\sqrt{3}\right)}{-2}=\dfrac{2}{-2}\cdot\dfrac{\left(1-\sqrt{3}\right)}{1}$ $=-\left(1-\sqrt{3}\right)$

Video about rationalisation:

http://
http://www.youtube.com/watch?v=gumXUv3vX_I

MORE EXAMPLES:

For the following problems the instruction is to rationalise the denominator, which means to write an equivalent expression for it that does not have any radicals in the denominator.

Example 1: $\frac{1}{\sqrt{3}}$

Solution: $\frac{1\sqrt{3}}{\sqrt{3}\sqrt{3}} = \frac{\sqrt{3}}{3}$

Example 2: $\sqrt{\frac{11}{6}}$

Solution:

Example 3: $\frac{2\sqrt{3}}{5\sqrt{2}}$

Solution:

Example 4: $\frac{1}{\sqrt[5]{2}}$

Solution:

Example 4: $\frac{-3}{5\sqrt[4]{2^3}}$

Solution:

Example 5: $\frac{6\sqrt{6}-6}{\sqrt{6}}$

Solution:

Example 6: $\frac{-5}{2\sqrt{5}}$

Solution:

Example 7: $\dfrac{1-\sqrt{2}}{\sqrt{2}}$

Solution:

Example 8: $\dfrac{9}{5\sqrt[7]{5^5}}$

Solution:

Example 9: Rationalise the following fractions:

a) $\dfrac{1}{\sqrt{2}+1}$

b) $\dfrac{3}{\sqrt{2}+\sqrt{3}}$

c) $\dfrac{-5}{\sqrt{3}-2}$

d) $\dfrac{4\sqrt{2}}{3\sqrt{2}-\sqrt{5}}$

e) $\dfrac{7}{\sqrt{11}-3}$

f) $\dfrac{-5}{\sqrt{6}+\sqrt{7}}$

3. LOGARITHMS

3.1. DEFINITIONS.

In its simplest form, a logarithm answers the question:

How many of *one number* do we multiply together to get *another number?*

Example:

How many 2s do we need to multiply together to get 8?

Answer: **2 × 2 × 2 = 8**, so we needed to multiply **3** times 2 to get 8.

So the logarithm of 8 with base 2 is 3.

Notice we are dealing with three numbers:

- the number we are multiplying (a "2")
- how many times to use it in a multiplication (3 times, which is the logarithm)
- The number we want to get (an "8")

How to write it

We would write "the number of 2s you need to multiply to get 8 is 3" as

$$\log_2(8) = 3$$

So these two things are the same:

$$\underbrace{2 \times 2 \times 2}_{3} = 8 \quad \leftrightarrow \quad \log_2(8) = 3$$

base

The number we are multiplying is called the "base", so we can say:

- "the logarithm with base 2 of 8 is 3"
- or "log base 2 of 8 is 3"
- or "the base-2 log of 8 is 3"

More Examples

Example: What is $\log_5(625)$ ... ?

We are asking "how many 5s need to be multiplied together to get 625?"

5 × 5 × 5 × 5 = 625, so we need 4 of the 5s

Answer: $\log_5(625) = 4$

Example: What is $\log_2(64)$ ... ?

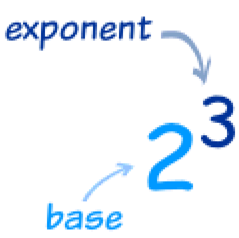
We are asking "how many 2s need to be multiplied together to get 64?"

2 × 2 × 2 × 2 × 2 × 2 = 64, so we need 6 of the 2s

Answer: $\log_2(64) = 6$

Exponents

Logarithms tell you what the exponent is!



The **exponent** of a number says **how many times** to use the number in a multiplication.

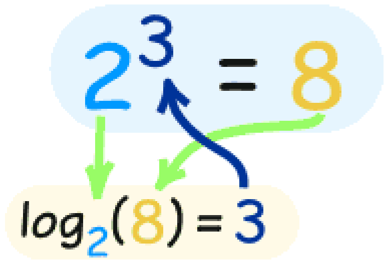
In this example: **2^3 = 2 × 2 × 2 = 8**

(2 is used 3 times in a multiplication to get 8)

Logarithms answer the question "what exponent produced this?":

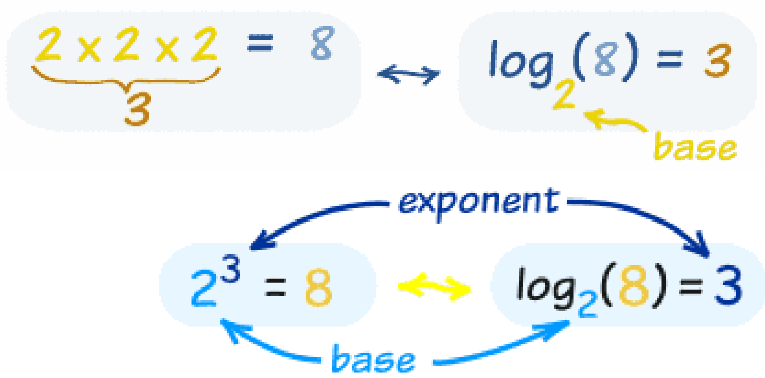
$$2^{?} = 8$$

And they answer it like this:



So this:

is also this:



So the logarithm answers the question:

What exponent do we need

(for one number to become another number) **?**

__Example__: What is $\log_{10}(100)$ ... ?

$10^2 = 100$, so to make 10 into 100 we need an exponent of 2

Answer: $\log_{10}(100) = 2$

__Example__: What is $\log_3(81)$ ... ?

$3^4 = 81$, so to make 3 into 81 we need an exponent of 4.

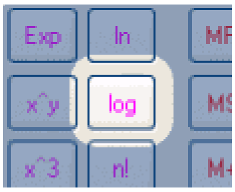
Answer: $\log_3(81) = 4$

Common Logarithms: Base 10

Sometimes you will see a logarithm written **without** a base, like this:

log(100)

This usually means that the base is actually 10.



This is called a "common logarithm". Engineers love to use it.

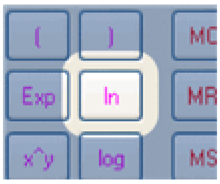
On a calculator it is the "log" button.

It is how many times you need to use 10 in a multiplication, to get the desired number.

Example: **log(1000) = $\log_{10}$(1000) = 3**

Natural Logarithms: Base "e"

Another base that is often used is e (Euler's Number) which is approximately 2.71828.



This is called a "natural logarithm". Mathematicians use this one a lot.

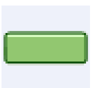
On a calculator it is the "ln" button.

It is how many times you need to use "e" in a multiplication, to get the desired number.

Example: **ln(7.389) = $\log_e$(7.389) ≈ 2**

Because **2.71828^2 ≈ 7.389**

Negative Logarithms



Negative? But logarithms deal with multiplying. What could be the opposite of multiplying? **Dividing!**

A negative logarithm means how many times **to divide** by the number.

We could have just one division:

Example: What is $\log_8(0.125)$ ... ?

Well, 1 ÷ 8 = 0.125, so $\log_8(0.125) = -1$

To put it another way:

Example: What is $\log_5(\mathbf{0.008})$ ... ?

$\mathbf{1} \div \mathbf{5} \div \mathbf{5} \div \mathbf{5} = \mathbf{5}^{-3}$, so $\log_5(\mathbf{0.008}) = -3$

To put it another way:

It All Makes Sense

Multiplying and Dividing are all part of the same simple pattern.

Let us look at some Base-10 logarithms as an example:

	Number	How Many 10s	Base-10 Logarithm
	.. etc..		
	1000	1 × 10 × 10 × 10	$\log_{10}(1000) = 3$
10x Larger	100	1 × 10 × 10	$\log_{10}(100) = 2$
	10	1 × 10	$\log_{10}(10) = 1$
	1	1	$\log_{10}(1) = 0$
10x Smaller	0.1	1 ÷ 10	$\log_{10}(0.1) = -1$
	0.01	1 ÷ 10 ÷ 10	$\log_{10}(0.01) = -2$
	0.001	1 ÷ 10 ÷ 10 ÷ 10	$\log_{10}(0.001) = -3$
	.. etc..		

If you look at the table above, you will see that positive, zero or negative logarithms are really part of the same (fairly simple) pattern.

The Word

"Logarithm" is a word made up by Scottish mathematician John Napier (1550-1617), from the Greek word logos meaning "proportion, ratio or word" and arithmos meaning "number", which together makes "ratio-number" ...

SOME EXERCISES:

1.What is $\log_4(256)$?

2. What is $\log_5(0.0016)$?

3. What is $\log_3(729)$?

4. What is $\log_2(0.015625)$?

5. Write $1{,}024 = 2^{10}$ in logarithmic form.

6. Write $\log_4(0.0625) = -2$ in exponential form.

7. What is the value of ln(5) (Use the calculator).

8. What is the value of ln(0.25) (Use the calculator)

9. Using the calculator, work out: a) log 23.5 b) log 267 c) log 0.0456

10. Using the calculator, work out the following logarithms rounding to two decimal numbers:

a) Ln 3 b) Ln 23.7 c) Ln 0.5

11. Use the definition on logarithms to calculate:

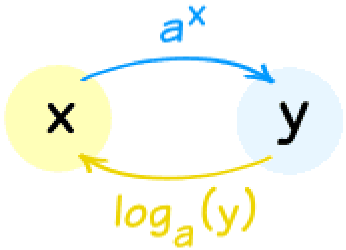
a) $\log_2 8$	c) $\log 1{,}000$	e) $\ln e^{22}$	g) $\log_4 16$
b) $\log_3 81$	d) $\log 0.0001$	f) $\ln e^{-4}$	h) $\log_4 0.25$

12. Use the definition of logarithms to calculate:

a) $\log_2 256$	c) $\log 1{,}000{,}000$	e) $\ln e^{2}$	g) $\log_7 343$
b) $\log_9 81$	d) $\log 0.00001$	f) $\ln e^{-14}$	h) $\log_4 0.0625$

3.2. WORKING WITH EXPONENTS AND LOGARITHMS

Exponents and Logarithms work well together because they "undo" each other (so long as the base "a" is the same):



They are "Inverse Functions"

So doing one, then the other, gets you back to where you started:

- Doing a^x, and then $\log_a(y)$, gives you x back again: $\boxed{\log_a(a^x) = x}$
- Doing the logarithm, then a^x , gives you x back again: $\boxed{a^{\log_a(x)} = x}$

It is a pity they are written *so differently*... it makes things look strange.

So it may help you to think of a^x as "up" and $\log_a(x)$ as "down":

- going up, then down, returns you back again: **down(up(x)) = x** , and
- going down, then up, returns you back again: **up(down(x)) = x**

Anyway, the important thing is that:

The Logarithmic Function can be "undone" by the Exponential Function.

(and vice versa)

As in this example:

Example, what is "x" in $\log_3(x) = 5$

We can use an exponent (with a base of 3) to "undo" the logarithm:

Example: Calculate "y" in $y=\log_4(1/4)$

3.3. PROPERTIES OF LOGARITHMS

One of the powerful things about logarithms is that they can **turn multiplying into adding**.

$$\log_a(m \times n) = \log_a m + \log_a n$$

"the log of a multiplication is the sum of the logs"

Why is that true?

To show you **why**, we have to use $a^{\log_a(x)} = x$ and $\log_a(a^x) = x$ again:

First, make **m** and **n** into "exponents of logarithms":

$$a^{\log_a(x)} = x$$

$$\log_a(m \times n) = \log_a(a^{\log_a m} \times a^{\log_a n})$$

Then use one of the Laws of Exponents: $x^m x^n = x^{m+n}$

$$= \log_a(a^{\log_a m + \log_a n})$$

Finally undo the exponents: $\log_a(a^x) = x$

$$= \log_a m + \log_a n$$

It is one of those clever things we do in mathematics which can be described as *"we cannot do it here, so let us go over **there**, then do it, then come back"*.
Have a look over that again so you understand why it happens. But the important thing is that **it does work**, and it is the result that matters.

Table of Properties

OK, we just used one of the Laws of Exponents that said $x^m \times x^n = x^{m+n}$, and we can use other Laws of Exponents as well!
Here are some of the most useful results for logarithms:

$\log_a(mn) = \log_a m + \log_a n$	**the log of a multiplication is the sum of the logs (as shown above)**
$\log_a(m/n) = \log_a m - \log_a n$	**the log of a division is the difference of the logs**
$\log_a(1/n) = -\log_a n$	**this just follows on from the previous "division" rule, because $\log_a(1) = 0$**
$\log_a(m^r) = r(\log_a m)$	**the log of m with an exponent r is r times the log of m**

Remember: the base "a" is always the same!

History:

Logarithms were very useful before calculators were invented... for example, instead of multiplying two large numbers, by using logarithms you could turn it into addition (much easier!)
And there were books full of Logarithm tables to help.

The Natural Logarithm and Natural Exponential Functions

When the base is **e** ("Euler's Number" = **2.718281828459**...) you get:

- The Natural Logarithm $\log_e(x)$, which is more commonly written $\ln(x)$
- The Natural Exponential Function e^x

And the same idea that one can "undo" the other is still true:

$$\ln(e^x) = x$$
$$e^{(\ln x)} = x$$

Changing the Base

What if you want to change the base of a logarithm?
Easy! Just use this formula:

$$\log_a x = \frac{\log_b x}{\log_b a}$$

"x goes up, a goes down"

Or another way to think of it is that $\mathbf{\log_b a}$ is like a "conversion factor" (same formula as above):

$$\log_a x = \log_b x \;/\; \mathbf{\log_b a}$$

So now you can convert from any base to any other base.

Another useful property is:

$$\log_a x = 1 \;/\; \log_x a$$

Did you see how "x" and "a" swap positions?

Example: Calculate $\log_2 8$

Example: Calculate $\log_4 22$

Solved example: Calculate $\log_5 125$

$$\log_5 125 = \ln 125 / \ln 5 = 4.83.../1.61... = 3 \text{ (exactly)}$$

I happen to know that 5 × 5 × 5 = 125, (5 is used **3** times to get 125), so I expected an answer of **3**, and it worked!

EXERCISE: Calculate the value of these logarithms:

a) $\log_4 32$	c) $\log_2 100$	e) $\log_{22} 4$
b) $\log_2 32$	d) $\log_5 32$	f) $\log_2 304$

REAL WORLD USAGE

Logarithms are used a lot in the real world. You may have heard of these:

- **Earthquakes**

The magnitude of an earthquake is a Logarithmic scale. The famous "Richter Scale" uses this formula:

$$M = \log_{10} A + B$$

Where **A** is the amplitude (in mm) measured by the

Seismograph and **B** is a distance correction factor.
Nowadays there are more complicated formulas, but they still use a logarithmic scale.

- **Sound**

Loudness is measured in Decibels (dB for short);

Loudness in dB = $10 \log_{10} (p \times 10^{12})$

where **p** is the sound pressure.

- **Acidic or Alkaline**

Acidity (or Alkalinity) is measured in pH:

$$pH = -\log_{10} [H^+]$$

where H^+ is the molar concentration of dissolved hydrogen ions.

Note: in chemistry [ ] means molar concentration (moles per litre).

More Examples

Example: Solve $2 \log_8 x = \log_8 16$

Exercise 1. Calculate the value of "x" in the following cases:

a) $3^2 = x$ b) $x^3 = 27$ c) $3^x = 1/3$

Exercise 2. Calculate the value of "x" in the following cases:

a) $2^{-3} = x$ b) $x^3 = 8$ c) $2^x = 1/4$

Exercise 3. Calculate in your mind.

a) $\log 100$ b) $\log 10$ c) $\log 0{,}001$

Exercise 4. Calculate in your mind. a) $\log_2 32$ b) $\log_2 1$ c) $\log_2 1/8$

Exercise 5. Calculate using the calculator rounding to four decimal numbers.

➔ a) log 23.5 b) log 267 c) log 0.0456

➔ a) L 3 b) L 23.7 c) L 0.5

Exercise 6. Work out the following logarithms using the properties of logarithms and the calculator:

a) $\log 3^{15}$ b) $\log \sqrt[7]{23}$ c) $\log (0{,}5^{30} \cdot 7^{23})$

Exercise 7. Use the symbol "=" or "$\neq$" in the following expressions:

a) $\log(7+5) \ldots \log 7 + \log 5$

b) $\log 5^2 \ldots 2 \log 5$

c) $\log \frac{6}{5} \ldots \log 6 - \log 5$

d) $\log \sqrt[3]{5} \ldots \log \frac{5}{3}$

Exercise 8. If log 5 = 0.6990, calculate:

a) $\log 2 =$

b) $\log 20 =$

Exercise 9. Reduce them to one logarithm.

a) $\log 5 + \log 6 - \log 2$

b) $2 \log 7 + 3 \log 5$

c) $3 \log a + 2 \log b - 5 \log c$

d) $2 \log x - 5 \log y + 3 \log z$

e) $\frac{1}{2} \log x + \frac{1}{3} \log y$

Exercise 10. Calculate the value of "x" in the following expressions with logarithms:

a) $\log_x 256 = -8$

b) $\log_2 x = \frac{2}{3}$

c) $\log_5 \sqrt{625} = x$

d) $\log_x 3 = 2$