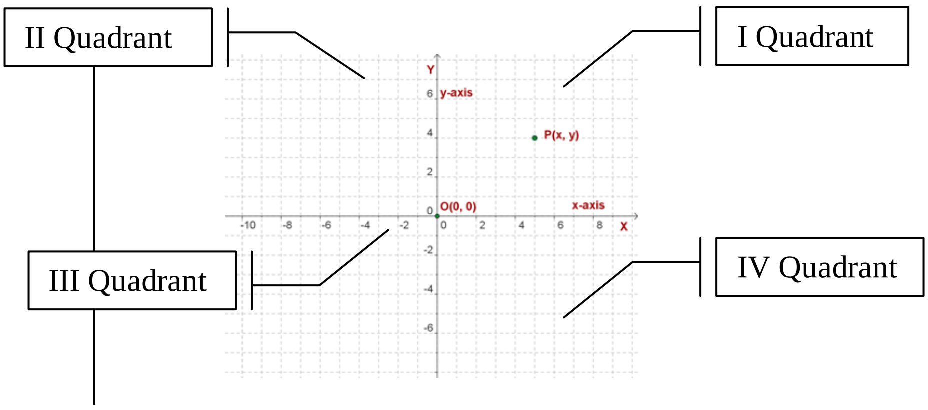
Unit 8. FUNCTION

1. COORDINATES IN THE PLANE

To represent points in the plane, we use two perpendicular straight lines. They are called the Cartesian axes or coordinate axes. The coordinate axes divide the plane into four equal parts called quadrants:



The horizontal axis is called the **x-axis**.

The vertical axis is called the **y-axis**.

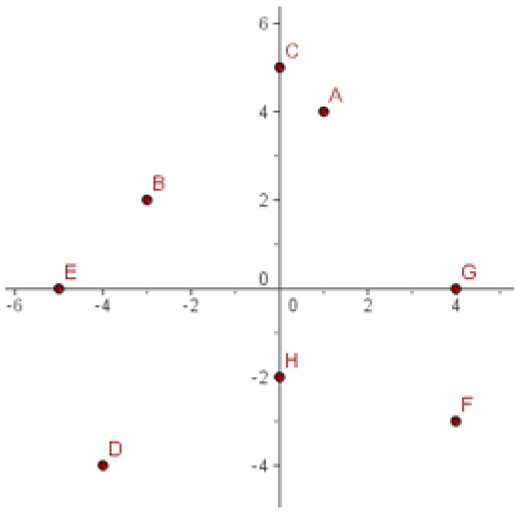
Point O, where the two axes intersect, is called the origin.

The coordinates of a point, P, are represented by **(x, y).**

PLOTTING POINTS

Solved Exercise: Plot the following points:

A(1, 4), B(-3, 2), C(0, 5), D(-4, -4), E(-5, 0), F(4, -3), G(4, 0), H(0, -2)



2. FUNCTION

A real function of real variables is any function, f, that associates a real number (image) to each element in a certain subset (domain).

$$\begin{aligned} f : D &\longrightarrow \mathbb{R} \\ x &\longrightarrow f(x) = y \end{aligned}$$

The subset which defines the function is called the **domain.**

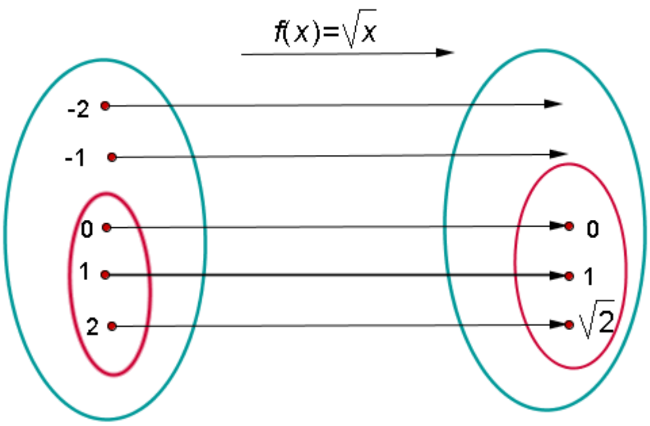
The number x belonging to the domain of the function is called the **independent variable**.

The number, y, associated by f to the value x, is called the **dependent variable**. The image of x is designated by f(x):

$$y = f(x)$$

The range of a function is the set of real values that the variable y or f(x) takes:

$$x \longrightarrow \sqrt{x}$$



Initial set Final set

DOMAIN AND RANGE

The **domain** is the set of elements that have an image:

$$D = \{x \in \mathbb{R} \,/\, \exists\ f(x)\}$$

The **range** is the set of the images.

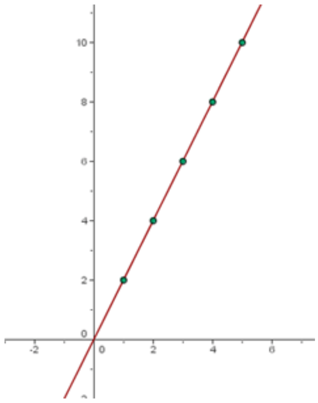
$$R = \{f(x) / x \in D\}$$

GRAPH OF A FUNCTION

If **f** is a real function, every pair (x, y) = (x, f(x)) determined by the function f corresponds to the Cartesian plane as a single point P(x, y) = =P(x, f(x)). The value of x must belong to the domain of the function.

The set of points belonging to a function is unlimited, and the pairs are arranged in a table of values which correspond to the points of the function. These values, on the Cartesian plane, determine points on the graph. Joining these points with a continuous line gives the graphical representation of the function.

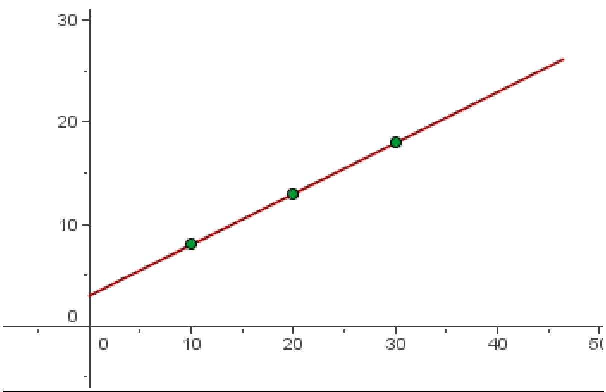
x	1	2	3	4	5
f(x)	2	4	6	8	10



Another example:

The price of a taxi ride is represented by: y = 3 + 0.5x, where x is the time in minutes of the ride (3 is the minimum fare).

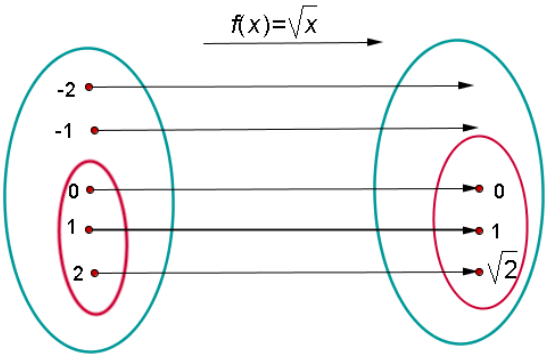
x	10	20	30
y= 3 + 0.5x	8	13	18



3. DOMAIN OF A FUNCTION

The domain is the set of elements that have an image.

D = {x∈ℝ/∃f(x)}



Initial set *Final set*

Study the Domain of a Function

EXAMPLE 1: Domain of a Polynomial Function

The domain is ℝ.

f(x)= x^2 − 5x + 6 **D = ℝ**

EXAMPLE 2: Domain of a Rational Function

The domain is ℝ, minus the values that would annul the denominator (there cannot be a number whose denominator is zero).

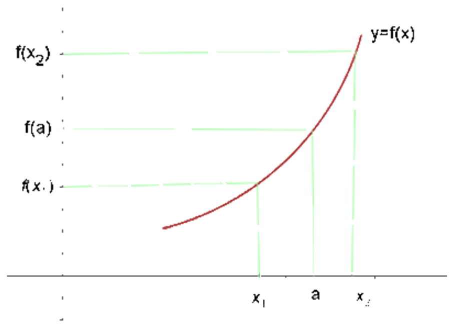
$$f(x) = \frac{2x-5}{x^2 - 5x + 6}$$

$x^2 - 5x + 6 = 0$ D = ℝ - {2, 3}

EXAMPLE 3: What do you think is the domain of $f(x)=\sqrt[3]{x}$? And the domain of $f(x)=\sqrt{x}$?

4. INCREASING AND DECREASING FUNCTIONS

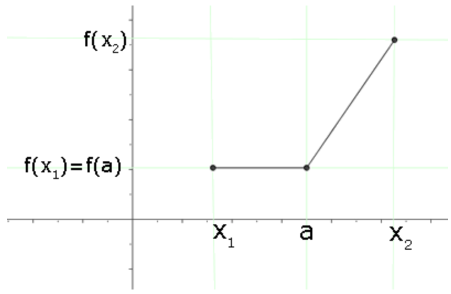
Strictly Increasing Function



$x > a \Rightarrow f(x) > f(a)$

$x < a \Rightarrow f(x) < f(a)$

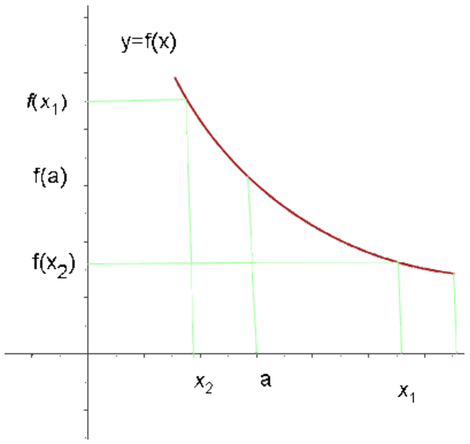
Increasing Function



$x > a \Rightarrow f(x) \geq f(a)$

$x < a \Rightarrow f(x) \leq f(a)$

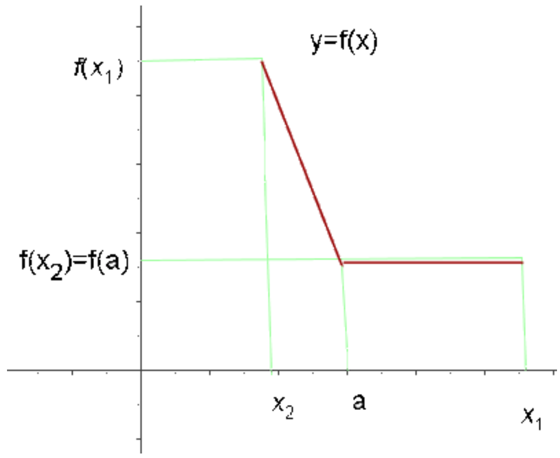
Strictly Decreasing Function



$x > a \Rightarrow f(x) < f(a)$

$x < a \Rightarrow f(x) > f(a)$

Decreasing Function



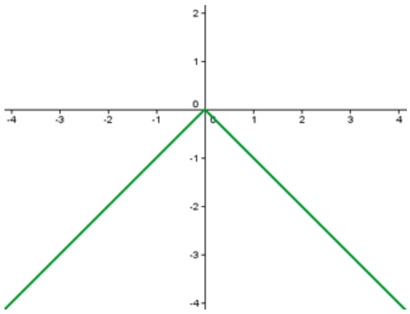
$x > a \Rightarrow f(x) \leq f(a)$

$x < a \Rightarrow f(x) \geq f(a)$

5. ABSOLUTE AND RELATIVE MAXIMUM AND MINIMUM

Absolute Maximum

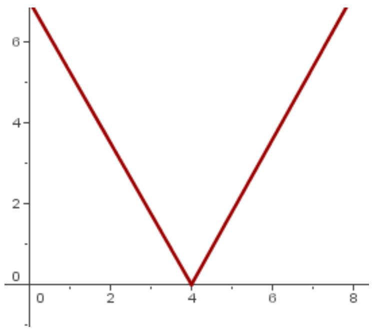
A function has its absolute maximum at x = a if the ordenate is greater than or equal to any point in the domain of the function.



a = 0

Absolute Minimum

A function has its absolute minimum at x = b if the ordenate is less than or equal to any point in the domain of the function.

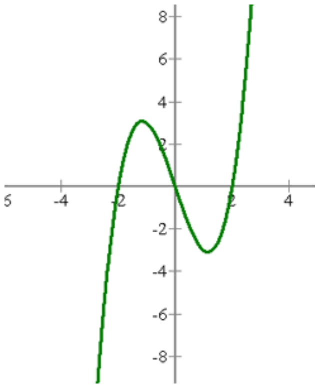


b = 0

Relative Maximum and minimum

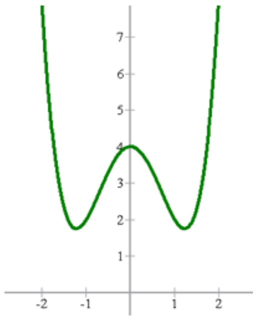
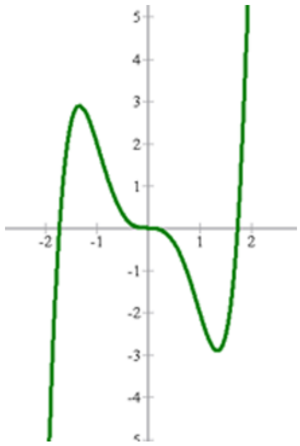
A function, f, has a relative maximum, at point a, if f(a) is greater than or equal to the points near the point a.

A function, f, has a relative minimum, at point b, if f(b) is less than or equal to the points close to point b.



a = 3.08 b = -3.08

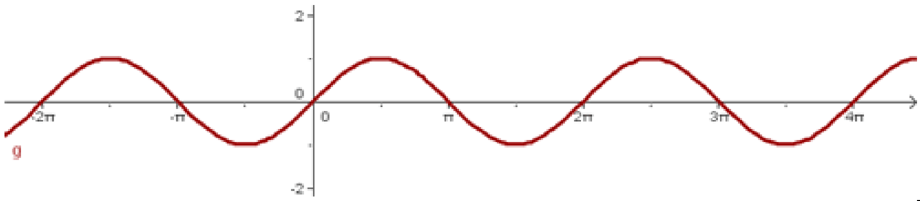
6. SIMMETRY: EVEN AND ODD FUNCTIONS

Even Function - Symmetry about the Vertical Axis	Odd Function - Symmetry about the Origin
The functions that are symmetrical about the vertical axis are called even functions. 	The symmetrical functions about the origin are called odd functions. 

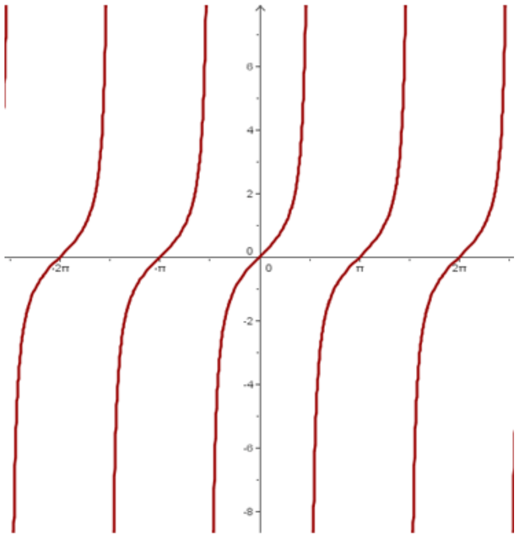
7. PERIODIC FUNCTIONS

A function, f(x), is **periodic** for period T, if it is verified for every integer (z):

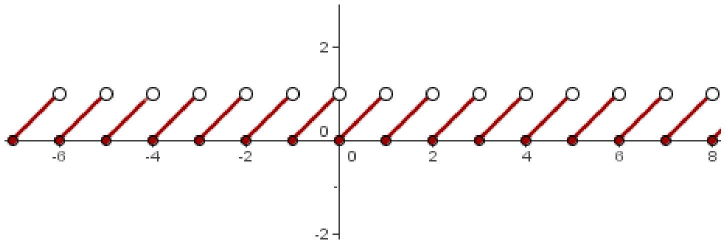
$$f(x) = f(x + zT)$$



T = …….

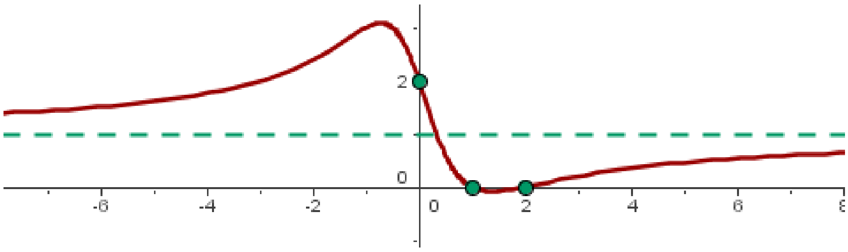


T = …….



T = …….

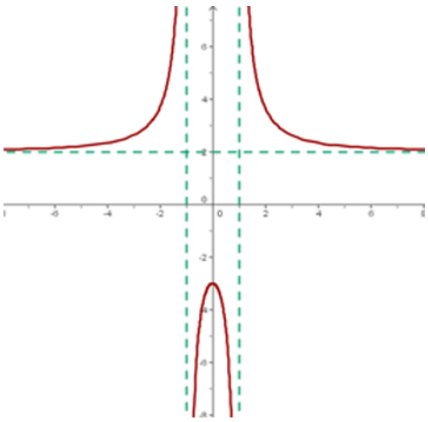
8. X- AND Y- INTERCEPTS



X-intercept: ………

Y- intercept: ……

9. ASYMPTOTES

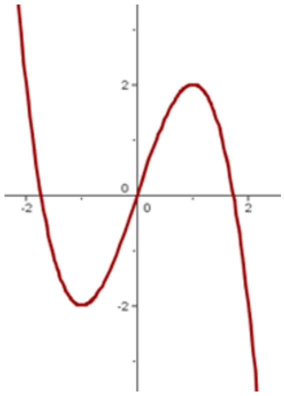


Horizontal Asymptotes: ………

Vertical Asymptotes:…………

EXERCISES

1. Study the characteristics of the following function:



Dom=
Range=
Increasing interval:
Decreasing interval:
Simmetry:

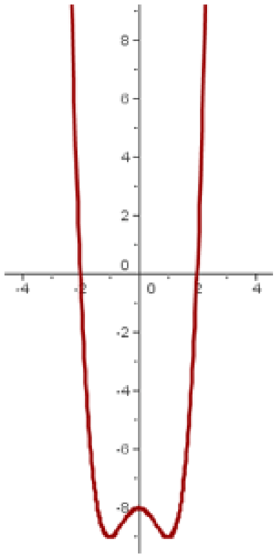
Period:

X-intercept:
Y-intercept:

Maximum:
Minimum:

Asymptotes:

2. Study the characteristics of the following function:



Dom=
Range=
Increasing interval:
Decreasing interval:
Simmetry:

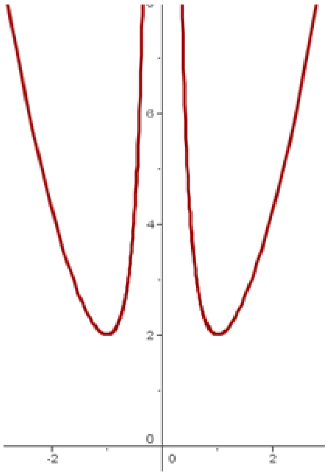
Period:

X-intercept:
Y-intercept:

Maximum:
Minimum:

Asymptotes:

3. Study the characteristics of the following function:



Dom=……………………………
Range= ……………………………
Increasing interval: ……………………
Decreasing interval: …………………
Simmetry: ……………………………

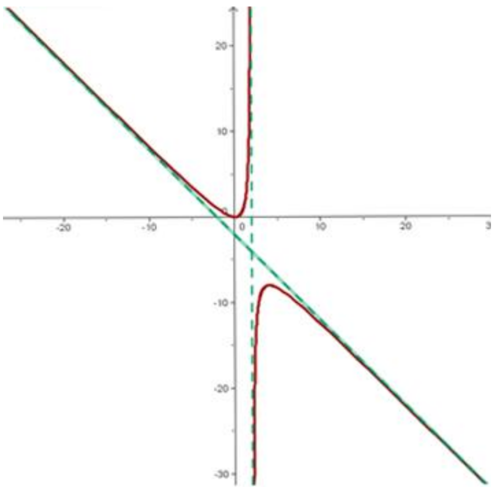
Period: ……………………………

X-intercept: ………………………
Y-intercept: ………………………

Maximum: ……………………………
Minimum: ……………………………

Asymptotes: …………………………
……………………………

4. Study the characteristics of the following function:



Dom=……………………………
Range= ……………………………
Increasing interval: ……………………
Decreasing interval: …………………
Simmetry: ……………………………

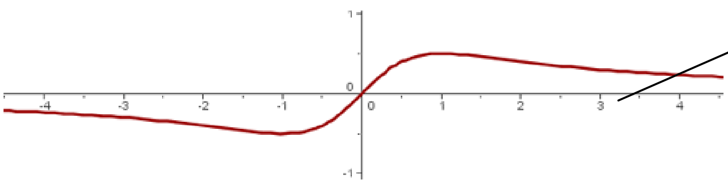
Period: ……………………………

X-intercept: ………………………
Y-intercept: ………………………

Maximum: ……………………………
Minimum: ……………………………

Asymptotes: …………………………
……………………………

5.Study the characteristics of the following function:



Dom=……………………………
Range= ……………………………
Increasing interval: ……………………
Decreasing interval: …………………
Simmetry: ……………………………

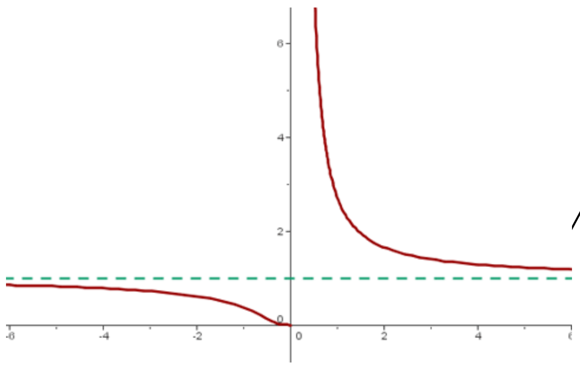
Period: ……………………………

X-intercept: ………………………
Y-intercept: ………………………

Maximum: ……………………………
Minimum: ……………………………

Asymptotes: …………………………
……………………………

7. Study the characteristics of the following function:



Dom=
Range=
Increasing interval:
Decreasing interval:
Simmetry:

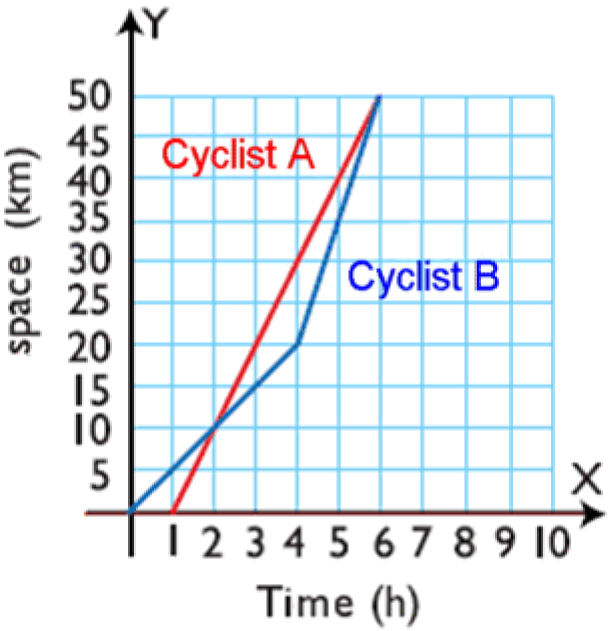
Period:

X-intercept:
Y-intercept:

Maximum:
Minimum:

Asymptotes:

8. The graph below shows the route of two cyclists. Analise them and answer: How far did they cycle? Did they start at the same time? Which cyclist travelled the fastest? Did they meet at any point in time?



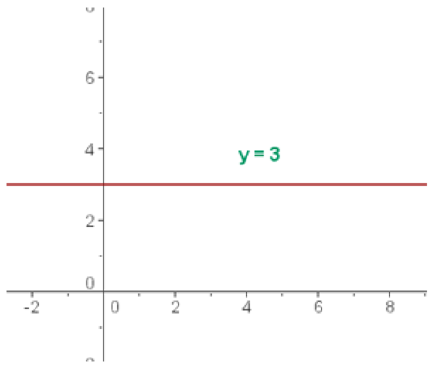
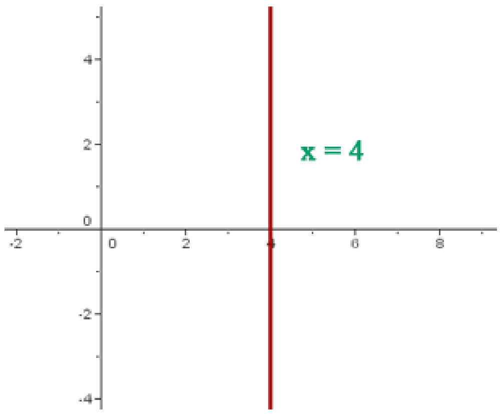
9. On a small island there are two taxi companies. The minimum fare for company A is €1.5, and they charge €0.35 for each kilometre. The minimum fare for company B is €2.5, and they charge €0.25 for each kilometre. Draw on a graph the functions representing the cost of a trip depending on the kilometre you travel with each company, and decide which company is cheaper to go on a certain journey.

10. TYPES OF FUNCTIONS

- Functions
  - Algebraic
    - Polynomial
      - Constant
      - Linear
      - Quadratic
    - Rational
    - Radical
    - Piecewise
  - Transcendental
    - Exponential
    - Logarithmic
    - Trigonometric

This year, we will focus on polynomial functions until the second degree although we will introduce some irrational functions.

10.1. CONSTANT FUNCTIONS

Horizontal Lines	Vertical Lines
The equation of a constant function is: $y = b$ The criterion is given by a real number. The slope is 0. The graph is a horizontal line parallel to the x-axis 	The lines parallel to the y-axis are not functions. The equation of a vertical line is: $x = a$. 

10.2. LINEAR FUNCTIONS

The equation of a linear function is:

$$y = mx + b$$

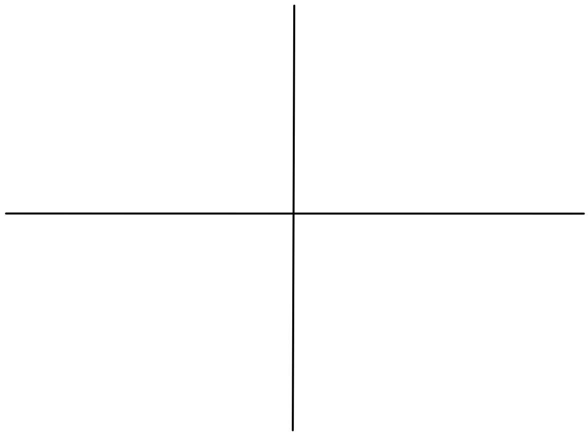
Its graph is an oblique straight line, which is defined by two points of the function.

Example: **$y = x + 4$ → $m = 1$ $b = 4$**

Its **graph** is an **oblique straight line**,

which is defined by two points of the function.

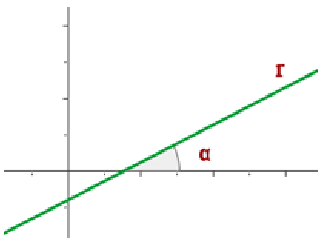
x	0	-4
y = x + 4	4	0



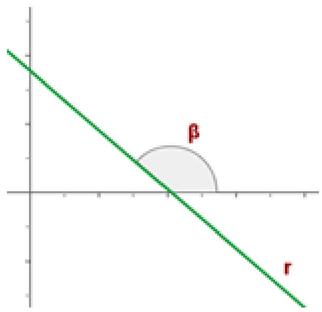
m is the slope of the straight line

The slope is the inclination of the line with respect to the x-axis.

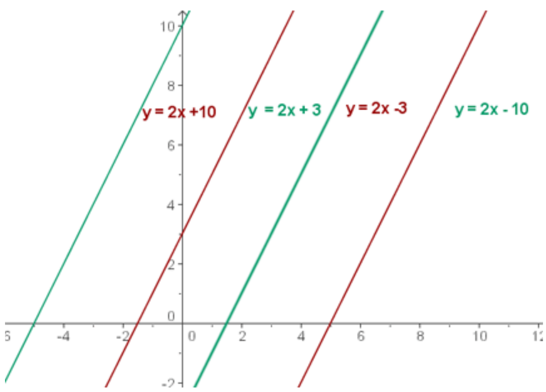
If **m > 0**, the **function** is **increasing** and the **angle** of the line with the positive x-axis is **acute.**



If **m < 0**, the **function** is **decreasing** and the **angle** between the line with the positive x-axis is **obtuse**.



Two parallel lines have the same slope. **Example**: m=2



Look at these straight lines:
Do you know what **n** indicates?

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EXERCISE 10. Graph the following linear functions, indicating the slope and y-intersection.

a) $Y = 2x + 1$

b) $Y = -3/4\ x - 1$

10.3. QUADRATIC FUNCTIONS

The equation of a **quadratic function** is:

$$y = ax^2 + bx + c$$

Its **graph** is a **parabola**.

Graphical Representation of the Parabola

A parabola can be built from these points:

1.- Vertex

$$x_v = \frac{-b}{2a} \qquad y_v = f\left(\frac{-b}{2a}\right) \qquad V\left(\frac{-b}{2a}, f\left(\frac{-b}{2a}\right)\right)$$

The **axis of symmetry** passes through the **vertex** of the parabola.

The equation of the **axis of symmetry** is: $x = \frac{-b}{2a}$

2.- x-intercepts

For the intercept with the x-axis, the **second coordinate** is always **zero**:

$$ax^2 + bx + c = 0$$

To find the x-intecepts, solve the resulting quadratic equation:

Two intercept points: $(x_1, 0)$ $(x_2, 0)$ if $b^2 - 4ac > 0$

One intercept point: $(x_1, 0)$ if $b^2 - 4ac = 0$

No intercept points if $b^2 - 4ac < 0$

3.- y-intercept

For the intercept with the y-axis, the **first coordinate** is always **zero**:

$f(0) = a \cdot 0^2 + b \cdot 0 + c = c$ $(0, c)$

Example: Graph the quadratic function $y = x^2 - 4x + 3$.

1. Vertex

$x_v = -(-4)/2 = 2$ $y_v = 2^2 - 4 \cdot 2 + 3 = -1$

$V(2, -1)$

2. x-intercepts

$x^2 - 4x + 3 = 0$

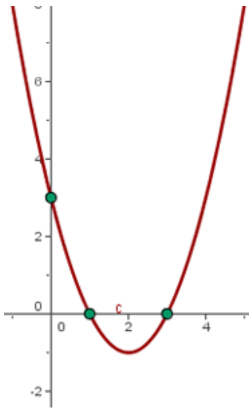
$$x = \frac{4 \pm \sqrt{16 - 12}}{2} = \frac{4 \pm 2}{2} \quad \begin{matrix} x_1 = 3 \\ x_2 = 1 \end{matrix}$$

$(3, 0)$ $(1, 0)$

3. y-intercept

$(0, 3)$

SOLUTION:



EXERCISE 11. Graph the following quadratic functions, indicating the vertex:

a) $y = x^2$

b) $y = x^2 + 1$

c) $y = x^2 - 2$

d) Could you conclude something from the previous graphs?................
………………………………………………………………………………………………

e) $y = (x - 2)^2$

f) Could you conclude something from the previous graph again?...
………………………………………………………………………………………………

10.4. HYPERBOLA

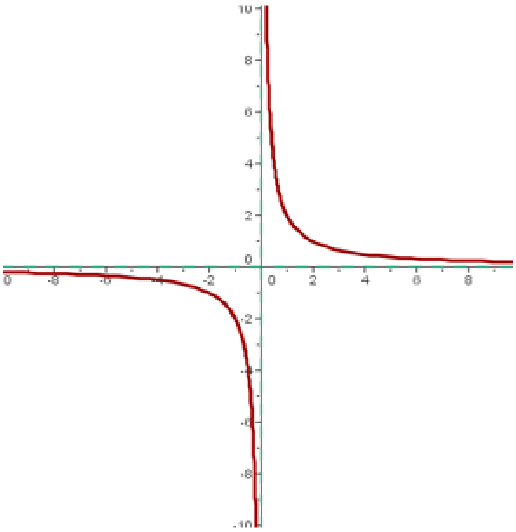
Hyperbolas are also the graphs of functions $y = \dfrac{ax+b}{cx+d}$.
The simplest **hyperbola** is represented with the equation

$$y = \frac{k}{x}.$$

Its **asymptotes** are the **axes**.

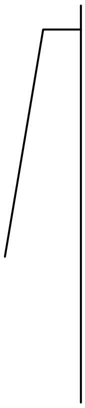
The **center** of the **hyperbola**, which is where the asymptotes intersect, is the **origin**.

$$f(x) = \frac{2}{x}$$



EXAMPLE: VERTICAL TRANSLATION

$$y = \frac{2}{x} + 3$$



Conclusions:
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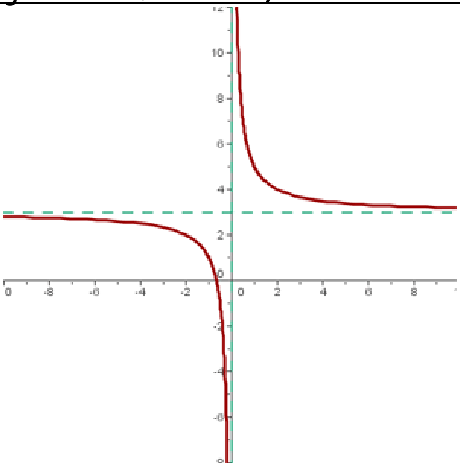
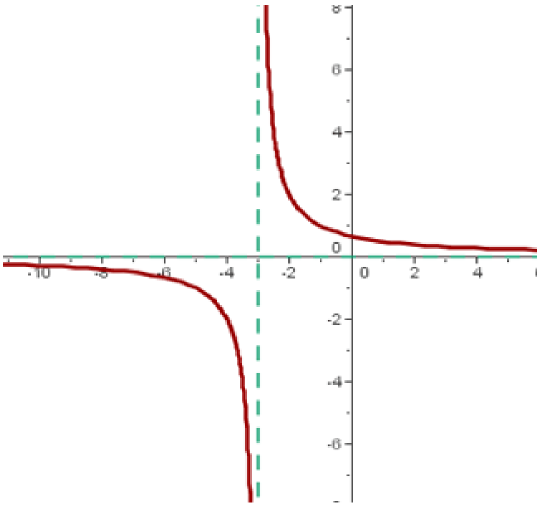


EXAMPLE: HORIZONTAL TRANSLATION

$$y = \frac{2}{x+3}$$



Conclusions:
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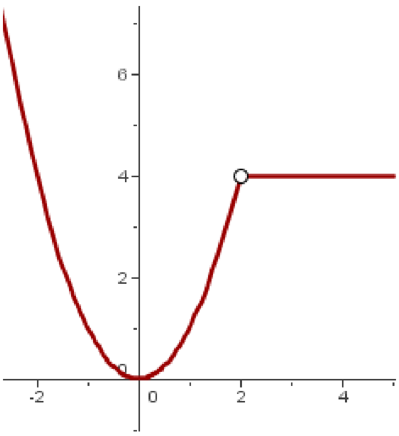
10.5. PIECE FUNCTIONS

Piecewise functions are functions defined by different criteria according to the intervals being considered.

Example: Graph the following piece function:

$$y = \begin{cases} x^2 & \text{if } x < 2 \\ 4 & \text{if } x > 2 \end{cases}$$



EXERCISES

UNIT 9. FUNCTIONS.

ABOUT LINEAR FUNCTION:

1. $Y = 3$

2. $X = 2$

3. $y = 2x - 1$

4. $y = -2x - 1$

5. $y = ½x - 1$

6. $y = -\frac{1}{2}x - 1$

7. Graph the following functions based on the information given:

a) The line has a slope of −3 and its y-intercept is −1.

b) The line has a slope of 4 and passes through point (−3, 2).

c) The line passes through points A (−1, 5) and B (3, 7).

d) The line passes through point P (2, −3) and is parallel to the line

$y = -x + 1$.

8. We can buy three pounds of squid at the market for $18. Determine the equation and represent the function that defines the cost of squid based on weight.

9. A car rental charge is $100 per day plus $0.30 per mile travelled. Determine the equation of the line that represents the daily cost by the number of miles travelled and graph it. If a total of 300 miles was travelled in one day, how much is the rental company going to receive as payment?

10. It has been observed that a particular plant's growth is directly proportional to time. It measured 2 cm when it arrived at the nursery and 2.5 cm exactly one week later. If the plant continues to grow at this rate, determine the function that represents the plant's growth and graph it.

ABOUT QUADRATIC FUNCTIONS:

11. Graph the following quadratic functions:

1. $y = -x^2 + 4x - 3$

2. $y = x^2 + 2x + 1$

3. $y = x^2 + x + 1$

12. A quadratic function has an equation in the form $y = x^2 + ax + a$ and passes through point (1, 9). Calculate the value of a.

13. From the graph of function $f(x) = x^2$, graph the following translations:

1. $y = x^2 + 1$

2. $y = x^2 - 3$

3. $y = (x + 1)^2$

4. $y = (x - 1)^2$

5. $y = (x - 1)^2 + 2$

14. The quadratic equation $y = ax^2 + bx + c$ passes through points (1,1), (0, 0) and (−1,1). Calculate the value of a, b and c.

ABOUT HYPERBOLAS.

15. Graph the Following Rational Functions:

a) **f(x) = 6/x**

b) $f(x)=\frac{6}{x}+3$

c) $f(x)=\frac{6}{x}-3$

d) $f(x)=\frac{2}{x}$

e) $f(x)=\frac{-1}{x}$

f) $f(x)=\frac{1}{x-2}$

16. Graph the following piecewise functions:

a) $f(x)=\begin{cases}2x+4 & \text{if} \quad x>0\\ 4-2x & \text{if} \quad x<0\end{cases}$

b)
$$f(x)=\begin{cases} 1 & if \quad x\le 1 \\ x & if \quad 1<x\le 3 \\ -x+6 & if \quad 3<x\le 6 \\ 0 & if \quad 6<x \end{cases}$$

c)
$$f(x)=\begin{cases} x^2 & if \quad x<2 \\ 1 & if \quad x=2 \\ 4 & if \quad x>2 \end{cases}$$

ABOUT CHARACTERISTICS OF FUNCTIONS.

17. Exercise 14. Page 155.

18. Exercise 15. Page 155.

19. Exercise 16. Page 156.

20. Exercise 39. Page 158.

21. Exercise 40. Page 158.

22. Exercise 41. Page 159.

23. Exercise 42. Page 159.

24. Exercise 44. Page 159.

25. Exercise 64. Page 159.

26. Exercise 65. Page 160.

27. Exercise 66. Page 160.