Unit 7. SYSTEMS OF LINEAR EQUATIONS

1. INTRODUCTION: SYSTEM OF EQUATIONS

A **system of equations** is any group of equations with common solutions.

A **solution** to a system of equations is a solution that works in every equation in the group. For example, in the following system of equations:

$$\left.\begin{array}{r} 2x-5y=-3 \\ -x+2y=3 \end{array}\right\}$$

the solution is x = -9, y = -3 because these values "work" in both equations.

However, x = -3, y = 0 is not a solution for the system, even though it is a solution for the second equation.

In this unit we will discuss four methods of solving systems of equations: the **graphical** method, **substitution**, **equalization** (in Spanish *"igualación"*, in the English-speaking world it is not studied as a separate method), and **elimination** (in Spanish "*reducción*").

<u>Solved example:</u> Check that x = 3 and y = 2 is a solution of the following system:

$$\left.\begin{array}{r} 4x+7y=26 \\ 6x-5y=8 \end{array}\right\} \Rightarrow \begin{array}{l} 4\cdot 3+7\cdot 2=12+14=26 \\ 6\cdot 3-5\cdot 2=18-10=8 \end{array}$$

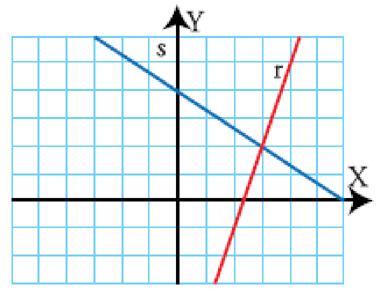
<u>Example</u>: Check if x = 2 and y = -1 is a solution of the following system:

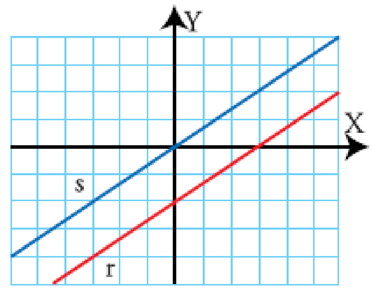
$$\left.\begin{array}{r} 2x-5y=9 \\ -x+2y=0 \end{array}\right\}$$

2. The Graphical Method

Think about this:

- **In which point do these straight lines cross?**
- **Have they got any point in common?**
- **How are these straight lines?**





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The **graphical method** consists of graphing every equation in the system and then using the graph to find the coordinates of the point(s) where the graphs intersect. **The point of intersection is the solution.**

Solved example: Use the graphical method to solve the system:

$$\left.\begin{aligned} x + y &= 4 \\ 2x - y &= -1 \end{aligned}\right\}$$

1st Step: Draw the first straight line corresponding to the first equation.

$$x + y = 4 \Rightarrow y = 4 - x$$

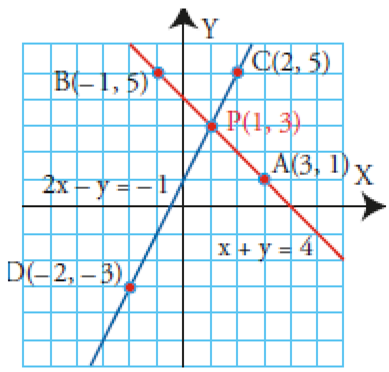
x	y = 4 – x
–2	6
–1	5
0	4
1	3
2	2

2nd Step: Draw the second straight line corresponding second equation.

$$2x - y = -1 \Rightarrow y = 2x + 1$$

x	y = 2x + 1
–2	–3
–1	–1
0	1
1	3
2	5

3th Step: The solution is the point of intersection of these straight lines: x = 1, y = 3.

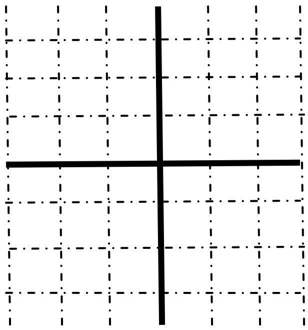


Example: Use the graphical method to solve the following system of equations..

$$\left.\begin{aligned} x - y &= -2 \\ -2x + y &= 3 \end{aligned}\right\}$$

Note: Graph both equations very precisely. If you don't graph carefully, your point of intersection will be way off.

- Draw the "x" and "y" axes.



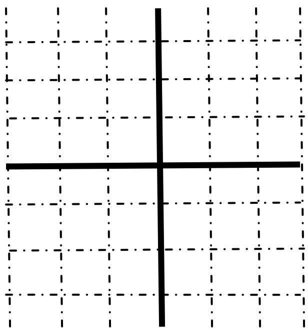
- Graph the first equation, it's a straight line: $x-y=-2$

- Graph the second equation, it's another straight line: $-2x+y=3$

The solution is the point of intersection.

<u>Example</u>: Use the graphic method to solve the following system of linear equations:

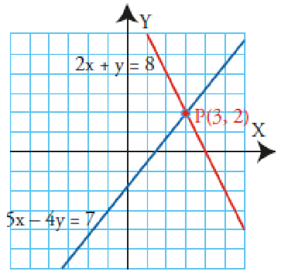
$$\left.\begin{array}{r} 2x+y=4 \\ -x+y=1 \end{array}\right\}$$



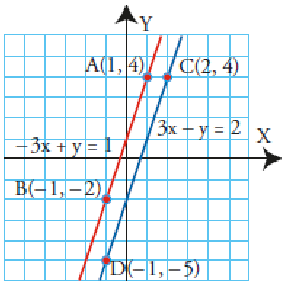
NUMBER OF SOLUTIONS IN A SYSTEM OF EQUATIONS:

A system of equations can be classified according to the number of solutions:

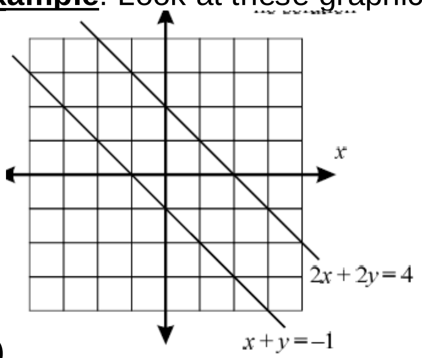
- **Compatible system**: if the system has got one solution, then the straight lines cut across each other in one point. Example:



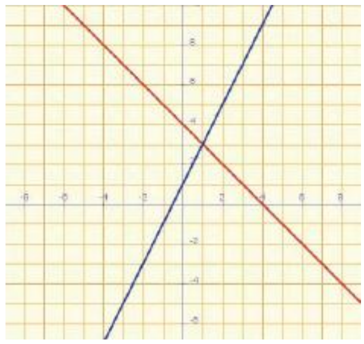
- **Incompatible system:** if the system hasn't got a solution, then the straight lines are parallel. Example:



Example: Look at these graphics and write the solutions of these systems of equations:



a)
SOLUTION:________________



b)
SOLUTION: ___________________

TO REVISE THIS THEORY, YOU CAN VISIT THIS USEFUL AND ENJOYABLE WEBSITE, WHERE YOU CAN WATCH SOME VIDEOS. COME ON, GIVE IT A TRY!

http://www.math-videos-online.com/systems-of-equations.html

Exercise 1. Solve the following systems using the graphic method. Decide if they are compatible or incompatible:

a) $\left.\begin{array}{r} x + y = 3 \\ 3x + y = 7 \end{array}\right\}$

b) $\left.\begin{aligned} -2x + y &= 3 \\ 2x - y &= 1 \end{aligned}\right\}$

c) $\left.\begin{aligned} 2x - y &= -1 \\ x + 2y &= 7 \end{aligned}\right\}$

d) $\left.\begin{aligned} x + 3y &= 2 \\ -x - 3y &= 2 \end{aligned}\right\}$

Exercise 2. The addition of two numbers is equal to 5. Write an equation expressing this condition and calculate five pairs of solutions. Draw this set of solutions.

Exercise 3. The subtraction of two numbers is 1. Write an equation expressing this condition and calculate five pairs of solutions. Draw the set of equations.

Exercise 4. Solve the following system of equations using the graphic method:

$$\left.\begin{array}{r} x - 2y = 1 \\ -x + 2y = 5 \end{array}\right\}$$

3. Substitution Method

The method of solving "by substitution" works by solving one of the equations (you choose which one) for one of the variables (you choose which one), and then using this in (or *plugging this back into*) the other equation, *substituting*[1] it for the chosen variable and solving the second equation. When you know the value of the second variable, you can easily calculate the first variable (*back-solve for it*).

Here is how it works. (We'll use the same systems that we used in the previous pages.)

Solved example: (substitution method)

$$\left.\begin{array}{l} 2x - 3y = -2 \\ 4x + y = 24 \end{array}\right\}$$

The idea here is to solve one of the equations for one of the variables, and replace that variable with this in the other equation (*plug it into* the other equation). It does not matter which equation or which variable you choose. There is no right or wrong choice; the answer will be the same either way. However, some choices may make the calculations easier than others.

For instance, in this case, we can see that the simplest way is probably to solve the second equation for "**y =**", since there is already an isolated **y** in the middle of the equation:

$$y = 24 - 4x$$

[1] When you **substitute** x **for** y, you **replace** y **with** x.

We could also solve **x** , but we would get fractions:

$$x = \frac{24 - y}{4}$$

Solving the first equation for either variable would also give me fractions. It wouldn't be "wrong" to make a different choice, but it would probably be more difficult. So it is easier to solve the second equation for y:

1st step: Work out the value of “y” in the second equation (because it’s the easiest one)

$$\left.\begin{array}{l} 2x - 3y = -2 \\ y = 24 - 4x \end{array}\right\}$$

2nd step: Now plug this into the first equation (substitute it for **y** in the first equation), and solve it for **x**:

2x – 3(– 4x + 24) = –2
2x + 12x – 72 = –2
14x = 70
x = 5

Warning: If you substitute the "24 – 4x" expression into the same equation that you have used to solve for "**y =**", which in our example was 4x + y = 24, you get a true, but useless, statement:

4x + (–4x + 24) = 24
4x – 4x + 24 = 24
24 = 24

It is true that twenty-four equals twenty-four, but that doesn’t help us. So when using substitution, make sure you substitute into the other equation, or you'll just be wasting your time.

3rd step: Calculate the value of the other unknown. We can plug this **x**-value into either equation, and solve for **y**, but since we already have an expression for "**y =**", the simplest solution is to just plug it in there:

y = 24 – 4x
y = 24 – 4(5) = 24 – 20 = 4

Then the solution is (x, y) = (5, 4).

Solved examples. Visit these web pages to watch some videos which revise the substitution method:

http://video.google.com/videoplay?docid=-6836161716522364284#
http://www.ehow.com/video_4754360_solve-equations-substitution.html

Exercise 5. Solve the following system using the substitution method. Write if it is compatible or incompatible:

a) $\left.\begin{array}{r} 2x + y = 4 \\ 3x + 4y = 11 \end{array}\right\}$

b) $\left.\begin{array}{r} y = 3x \\ 3x + y = 12 \end{array}\right\}$

c) $\left.\begin{array}{r} 2x + 3y = -5 \\ x - 2y = 8 \end{array}\right\}$

d) $\left.\begin{array}{l} \dfrac{x}{2} + 3y = 11 \\ 2x - \dfrac{y}{3} = 7 \end{array}\right\}$

4. “Equalization” Method.

This method consists in:

1st step: Work out the value of the same variable in both equations. (You choose the variable)

2nd step: Make both expressions equal.

3th step: Solve this equation.

4th step: To calculate the value of the other variable, plug the first back into one of the equations, "substituting".

Example: Solve using the Equalization Method:

$$\left.\begin{aligned} 3x + y &= 5 \\ -4x + y &= -9 \end{aligned}\right\}$$

Solution:

a) Se despeja la incógnita **y** de las dos ecuaciones.	$y = 5 - 3x$ $y = 4x - 9$
b) Se igualan los valores obtenidos.	$5 - 3x = 4x - 9$
c) Se resuelve la ecuación resultante.	$-7x = -14$ $7x = 14$ $x = 2$
d) Se sustituye el valor obtenido en la ecuación más sencilla donde estaba despejada la otra incógnita.	$x = 2$ en $y = 5 - 3x$ $y = 5 - 3 \cdot 2$ $y = 5 - 6$ $y = -1$
Solución: $x = 2,\ y = -1$	

Exercise 6. Solve using the Equalization Method.

a) $\left.\begin{array}{l} x+y=1 \\ 3x-y=-9 \end{array}\right\}$

b)

$\left.\begin{array}{l} 3x-y=13 \\ 2x-y=18 \end{array}\right\}$

5. Elimination or addition method.

To learn the procedure to solve a system of equations using the elimination method, please visit these videos in the following websites:

http://www.youtube.com/watch?v=xB-oXaCoJoc

http://www.youtube.com/watch?v=ej25myhYcSg

Solved example: Solve the system of linear equations. $\left.\begin{array}{l} -2x+3y=8 \\ 3x-y=5 \end{array}\right\}$

- Multiply all the terms in the second equation by 3:

  -2x + 3y = 8
  9x - 3y = -15

- Add the two equations (you can also subtract them if it is easier):

  7x = -7

- Note: **y** has been eliminated (when the equations are added), hence the name: method of elimination (or addition). Now solve the above equation for **x**:

x = -1

- Replace **x** with -1 in the first equation:

  -2(-1) + 3y = 8

- Solve the above equation for **y:**

  2 + 3y = 8
  3y = 6
  y = 2

- Write the solution to the system as an ordered pair:

  (-1,2)

  Check the solution obtained:

  first equation: Left Side: -2(-1) + 3(2)= 2 + 6 = 8
  Right Side: 8
  second equation: Left Side: 3(-1)-(2)=-3-2=-5
  Right Side: -5

- Conclusion: The given system of equations is consistent, and it has the ordered pair which is shown below as a solution.
  (-1,2)

Exercise 6. Solve the following systems using the most suitable method and classify them:

a) $\left.\begin{array}{l} 3x + 2y = 23 \\ 5x - 2y = 17 \end{array}\right\}$

b) $\left.\begin{array}{l} 5x + 3y = 7 \\ 4x + 3y = 5 \end{array}\right\}$

c) $\left.\begin{array}{l} 2x + 3y = -4 \\ 5x - 6y = 17 \end{array}\right\}$

d) $\left.\begin{array}{l} 5x + y = 13 \\ 4x + y = 10 \end{array}\right\}$

e) $\left.\begin{array}{r} -2x - 3y = 11 \\ 5x - 4y = 30 \end{array}\right\}$

f) $\left.\begin{array}{r} 0{,}5x + y = \quad 1 \\ 0{,}25x - y = -0{,}25 \end{array}\right\}$

NOTE: if you want to practise more, you can visit this website on the Internet: http://www.emathematics.net/sistecuaciones.php?ejercicio=simple&def=find

6.PROBLEMS WITH SYSTEMS OF EQUATIONS.

Exercise 1. Calculate two numbers which add up to 51, knowing that one of them is twice as big as the other.

Exercise 2. In a garage there are cars and motorcycles adding up to 18 vehicles. How many motorcycles and cars are there?

Exercise 3. The admission fee at a museum is \$1.50 for children and \$4.00 for adults. On a certain day, 2,200 people visit the museum, and they pay a total of \$5,050. How many children and how many adults visited the museum?

Exercise 4. The air-mail rate for letters from the USA to Europe is 45 cents per half-ounce, and to Africa it is 65 cents per half-ounce. If Shirley paid \$18.55 to send 35 half-ounce letters abroad, how many did she send to Africa?

Exercise 5. The perimeter of an isosceles triangle is 65 meters, and each of the equal sides is twice as long as the unequal side. How long is every side?

Exercise 6. Three DVDs and two CDs cost € 12; 4 DVDs and 4 CDs cost € 18. Calculate how much every DVD and every CD costs.

Exercise 7. Calculate the equation of a straight line " $ax + by = 2$" which goes through points A(1,2) and B(3,7).

EXERCISES

UNIT 7. SYSTEM OF EQUATIONS.

1. Use the graphic method to solve the following systems of equations.

a) $\left.\begin{array}{r} 3x - y = 5 \\ 2x + 3y = -4 \end{array}\right\}$

b) $\left.\begin{array}{r} 2x + y = -6 \\ 3x - y = 1 \end{array}\right\}$

c) $\left.\begin{array}{r} 3x + y = 10 \\ 2x + 3y = 9 \end{array}\right\}$

2. Read the following summary to classify a system of equations considering its solutions:

Clasificación de los sistemas	Compatible determinado	Incompatible	Compatible indeterminado
Criterio	$\frac{a}{a'} \neq \frac{b}{b'}$	$\frac{a}{a'} = \frac{b}{b'} \neq \frac{c}{c'}$	$\frac{a}{a'} = \frac{b}{b'} = \frac{c}{c'}$
Interpretación gráfica	Rectas secantes	Rectas paralelas	Rectas coincidentes

According to this classification using the coefficients, classify the following systems of equations:

a) $\left.\begin{array}{l} 2x + y = 1 \\ 2x + y = -1 \end{array}\right\}$

b) $\left.\begin{array}{l} x + 2y = 3 \\ 2x + 4y = 6 \end{array}\right\}$

c) $\left.\begin{array}{l} 3x - y = -5 \\ x + 2y = -4 \end{array}\right\}$

d) $\left.\begin{array}{l} x + 3y = 7 \\ 3x + 9y = -5 \end{array}\right\}$

e) $\left.\begin{array}{l} -2x + y = -1 \\ 4x - 2y = 2 \end{array}\right\}$

f) $\left.\begin{array}{l} 2x - y = 9 \\ 3x - 5y = 10 \end{array}\right\}$

3. Use the most appropriate method to solve the following systems of equations:

a) $\left.\begin{array}{l} x + 2y = 0 \\ 3x + 7y = 1 \end{array}\right\}$

b) $\left.\begin{array}{l} y = -2x + 3 \\ y = 5x - 4 \end{array}\right\}$

c) $\left.\begin{array}{l} \dfrac{x}{3} + \dfrac{y}{2} = 5 \\ \dfrac{x}{2} - \dfrac{y}{4} = 1 \end{array}\right\}$

d) $\left.\begin{array}{l} x + 0{,}75y = 3 \\ x - 0{,}5y = 5 \end{array}\right\}$

e) $\left.\begin{array}{l} 2x + y = 3 \\ 3x - 4y = 10 \end{array}\right\}$

f) $\left.\begin{array}{l} x = 2y + 3 \\ 3x + 4y = 5 \end{array}\right\}$

g) $\left.\begin{array}{l} \dfrac{x}{3} = \dfrac{y}{4} \\ 2x + 3y = 9 \end{array}\right\}$

h) $\left.\begin{array}{l} \dfrac{x}{2} + \dfrac{y}{3} = 3 \\ 5x + 2y = 4x + 10 \end{array}\right\}$

i) $$\left.\begin{array}{l} \dfrac{x+2y}{5} = 3 \\ 2x + 5y - 8 = 4(y+1) \end{array}\right\}$$

j) $$\left.\begin{array}{l} 0{,}25x + 0{,}5y = 2 \\ 0{,}75x - 0{,}5y = 5 \end{array}\right\}$$

4. Write a system of equations whose solution is : a = 3, b = -1.

5. Calculate the value of "k" if we know that x=2, y = 1 is a solution of the system:

$$\left.\begin{array}{r} x + 2y = 4 \\ kx - \quad y = 9 \end{array}\right\}$$

6. In a bicycle factory they make two different models, bicycle A and bicycle B. Bicycle A has got 1 kilogram of steel and 3 kilograms of aluminum; on the other hand, bicycle B has got 2 kilograms of steel and 2 kilograms of aluminum. If the factory has got 240 kilograms of steel and 360 kilograms of aluminum, how many bicycles could the factory make of each model?

7. Calculate two numbers knowing that dividing the biggest one by the other we get 2 as quotient and 3 as remainder, and both numbers add up to 39.