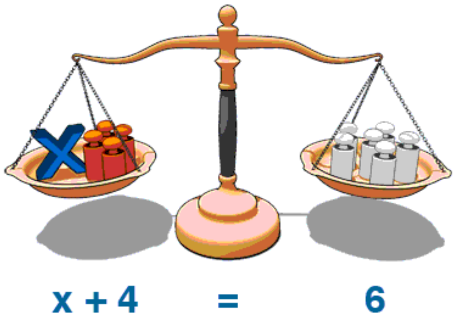
Unit 6. EQUATIONS.

1. BASICS OF ALGEBRA

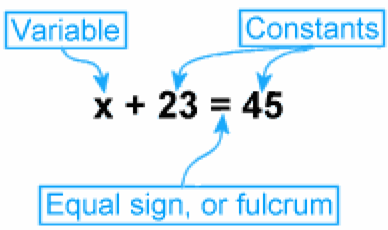
Algebra is a division of mathematics designed to help solve certain types of problems more quickly and easily. Algebra is based on the concept of unknown values called variables, unlike arithmetic, which is based entirely on known number values.

This lesson introduces an important algebraic concept known as the **Equation.** The idea is that an equation represents a pair of scales such as the ones shown below. Instead of keeping the scales balanced with weights, we use numbers, or constants. These numbers are called constants because they constantly have the same value. For example the number 47 always represents 47 units or 47 multiplied by an unknown number. It never represents another value.



The equation may also be balanced by a variable. ***A* variable** is an unknown number represented by any letter in the alphabet (most often x). The value of each variable must remain the same in each problem.

2. BASICS OF EQUATIONS.



The diagram on the left shows a basic equation. This equation is similar to problems which you may have done in ordinary mathematics such as:

__ + 16 = 30

You could easily guess that __ equals 14 or calculate 30 - 16 to find that __ equals 14.

In this problem __ stands for an unknown number; in an equation we use **variables**, or any letter in the alphabet.

When written algebraically, the problem would be:

$$x + 16 = 30$$

and the answer should be written:

$$x = 14$$

3. SOLVING EQUATIONS.

These equations can be solved relatively easily and without any formal method.

However, as you use equations to solve more complex problems, you will need an easier way to solve them.

Pretend you have a pair of scales like the ones shown on the left. On the right-hand pan there are 45 coins and on the left-hand pan there are 23 coins and an unknown amount of coins. The scales are balanced, therefore, we know that there must be an equal amount of weight on each side.

As long as the same operation (addition, subtraction, multiplication, etc.) is done to both sides, they will remain balanced. To find the unknown amount of coins on the left-hand pan, remove 23 coins from each pan. This action keeps the scales balanced and isolates the unknown amount. Since the weight (amount of coins) on both pans are still equal and the unknown amount is alone, we now know that the unknown amount of coins on the left-hand pan is the same as the remaining amount (22 coins) on the right-hand pan.

Because an equation represents a pair of scales, it can also be manipulated like one. The diagram below shows a simple equation and the steps to solve it.

Initial Equation / Problem	$x + 23$	$=$	45
Subtract 23 from each side	$x + 23 - 23$	$=$	$45 - 23$
Result / Answer	x	$=$	22

The diagram below shows a more complex equation. This equation has both a constant and a variable on each side. Again, to solve this you must keep both sides of the equation equal; perform the same operation on each side to get the variable "x" alone. The steps to solving the equation are shown below.

Initial Equation / Problem:	$x + 23$	$=$	$2x + 45$
Subtract x from each side	$x - x + 23$	$=$	$2x - x + 45$

Result	23	=	x + 45
Subtract 45 from each side	23 - 45	=	x + 45 - 45
Result	-22	=	x
Answer	x	=	-22

Take a look at the equation below. As you can see, after the variable is subtracted from the left and the constants are subtracted from the right, you are still left with 2x on one side.

Initial Equation / Problem	x + 23	=	3x + 45
Subtract x from each side	x - x + 23	=	3x - x + 45
Result	23	=	2x + 45
Subtract 45 from each side	23 - 45	=	2x + 45 - 45
Result	-22	=	2x
Switch the left and right sides of the equation	2x	=	-22

This means that the unknown number multiplied by two, equals -22. **But our end goal is to determine what x is**, not what 2x is! To find the value of x, we must "divide by the coefficient" as described below:

Imagine that three investors own an equal share in the company Example.com. The total worth of Example.com is $300,000. To determine what the share of each investor is, simply divide the total investment by 3:

$300,000 / 3 = $100,000

Thus, each investor has a $100,000 share.

We apply the same idea to finding the value of x. However, instead of dividing by the number of investors, we divide by the coefficient of the variable. Since we determined that the coefficient of x is 2, we divide each side of the equation by 2:

Divide by 2	2x / 2	=	-22/2
Result	1x	=	-11
Finally rewritten as	X =	=	-11

TO REVISE, LET'S WATCH THIS VIDEO:
http://www.mathplayground.com/howto_solvevariable.html

EXERCISE 1: Solve the following basic equations:

a) $4x + 12 = 6x - 8$

b) $6 + 3x = 4 + 7x - 2x$

c) $8x - 2x + 4 = 2x$

d) $4x + 3x - 4 = 3x + 8$

EXERCISE 2: Solve the following equations:

a) $3(x + 2) + 2x = 5x - 2(x - 4)$

b) $4 - 3(2x + 5) = 5 - (x - 3)$

c) $2(x - 3) + 5(x + 2) = 4(x - 1) + 3$

d) $5 - (2x + 4) = 3 - (3x + 2)$

Exercise 3. Solve in your mind:

a) $(x - 2)(x + 3) = 0$

b) $(2x + 1)(x - 4)(3x + 5) = 0$

Exercise 4. Solve the following equations with denominators.

a) $\frac{x-3}{4} = \frac{x-5}{6} + \frac{x-1}{9}$

b) $\frac{7-x}{2} = \frac{9}{2} + \frac{7x-5}{10}$

c) $\frac{x}{3} + 3x - \frac{x-2}{4} = \frac{1}{4} + x$

d) $\frac{x-1}{2} - \frac{x-2}{3} + \frac{10-3x}{5} = 0$

Exercise 5. Solve the following equations:

a) $12 - (7x + 5) = 4 - (5x + 2)$

b) $\frac{6x-1}{2} = \frac{x-1}{3} + \frac{4x+3}{2}$

c) $\frac{4-x}{5} = 2 - \frac{3x-2}{10}$

d) $\frac{3x}{2} - 2(x-3) - \frac{x-2}{4} = 5 + x$

e) $\frac{x-5}{2} - \frac{2x-3}{3} + \frac{10-x}{12} = 0$

4. CUADRATIC EQUATION

4.1. COMPLETE CUADRATIC EQUATION.

The Quadratic Formula uses the "a", "b", and "c" from **"ax^2 + bx + c"**, where "a", "b", and "c" are just numbers; they are the "numerical coefficients". In a complete cuadratic equation the numbers a, b and c are not zero.

The Formula is derived from the process of completing the square, and is formally stated as:

For $ax^2 + bx + c = 0$, the value of x is given by:		$x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

DON'T FORGET:

For the Quadratic Formula to work, you must have your equation arranged in the form "(quadratic) = 0". Also, the "2a" in the denominator of the Formula is underneath everything above, not just the square root. And it's a "2a" under there, not just a plain "2". Make sure that you are careful not to forget the square root or the "plus/minus" in the middle of your calculations, or I can guarantee that you will forget to "put them back" on your test, and you'll be in trouble. Remember that "b2" means "the square of ALL of b, including its sign", so don't leave b2 being negative, even if b is negative, because the square of a negative is a positive.

In other words, don't be sloppy and don't try to take shortcuts, because it will only hurt you in the long run. Trust me on this!

IF YOU WANT TO MEMORISE THE CUADRATIC FORMULA SINGING, HERE IS A FUNNY WAY:

http://www.youtube.com/watch?v=H_7lNT9oDzI

OR

http://www.youtube.com/watch?v=IvXgFLV2gOk

Here are some **examples** of how the Quadratic Formula works:

EXAMPLE 1: Solve $x^2 + 3x - 4 = 0$

This quadratic can factored:

$x^2 + 3x - 4 = (x + 4)(x - 1) = 0$

...so I already know that the solutions are $x = -4$ and $x = 1$. How would my solution look in the Quadratic Formula? Using $a = 1$, $b = 3$, and $c = -4$, my solution looks like this:

$$\begin{aligned} x &= \frac{-(3) \pm \sqrt{(3)^2 - 4(1)(-4)}}{2(1)} \\ &= \frac{-3 \pm \sqrt{9+16}}{2} = \frac{-3 \pm \sqrt{25}}{2} \\ &= \frac{-3 \pm 5}{2} = \frac{-3-5}{2}, \frac{-3+5}{2} \\ &= \frac{-8}{2}, \frac{2}{2} = -4,\ 1 = x \end{aligned}$$

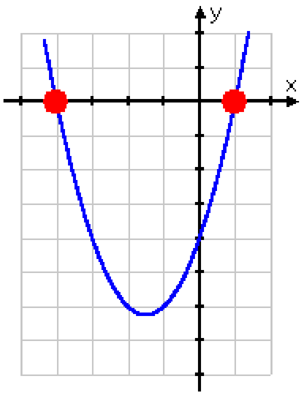
Then, as expected, the solution is $x = -4$, $x = 1$.

CORRESPONDENCE AMONG SOLUTIONS OF AN EQUATION AND GRAPH.

Suppose you have $ax^2 + bx + c = y$, and you are told to find values of x for $y = 0$.

The corresponding x values are the x-intercepts of the graph (the x-intercepts are the points where the graph crosses the x-axis).

So solving $ax^2 + bx + c = 0$ for x means that you are trying to find x-intercepts. Since there were two solutions for $x^2 + 3x - 4 = 0$, there must be two x-intercepts on the graph. Graphing, we get the curve below:



As you can see, the x-intercepts match the solutions, crossing the x-axis at $x = -4$ and $x = 1$.

This shows the connection between graphing and solving: When you are solving "(quadratic) = 0", you are finding the x-intercepts of the graph. This can be useful if you have a graphing calculator, because you can use the Quadratic Formula (when necessary) to solve a quadratic, and then use your graphing calculator to make sure that the displayed x-intercepts have the same values as the solutions provided by the Quadratic Formula.

Remember: The "solutions" of an equation are also the x-intercepts of the corresponding graph.

EXAMPLE 2: Solve x(x – 2) = 4. Round your answer to two decimal places.

There is a problem: I cannot apply the Quadratic Formula at this point, and I cannot factor it either, because on the right we have 4, not 0, so we cannot claim that "$x = 2$, $x - 2 = 2$". This is *not* how "solving by factoring" works.

I *must* first rearrange the equation in the form "(quadratic) = 0", whether I'm factoring or using the Quadratic Formula. First I have to do the multiplication on the left-hand side, and then I have to move the 4 over:

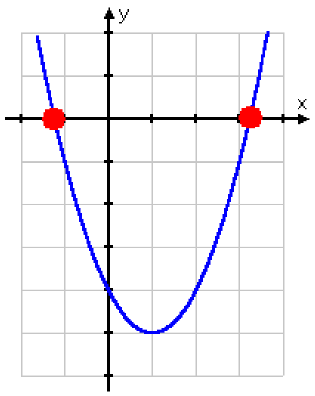
$$x(x-2)=4$$
$$x^2-2x=4$$
$$x^2-2x-4=0$$

We can use the Quadratic Formula; in this case, a = 1, b = –2, and c = –4:

$$x=\frac{-(-2)\pm\sqrt{(-2)^2-4(1)(-4)}}{2(1)}$$
$$=\frac{2\pm\sqrt{4+16}}{2}=\frac{2\pm\sqrt{20}}{2}=\frac{2\pm\sqrt{4}\sqrt{5}}{2}$$
$$=\frac{2\pm2\sqrt{5}}{2}=\frac{2\left(1\pm\sqrt{5}\right)}{2(1)}=1\pm\sqrt{5}$$
$$\approx -1.236068,\ \ 3.236968=x$$

Then the answer is: x = –1.24, x = 3.24, rounded to two places.

For reference, here's what the graph looks like:



- **EXAMPLE 3. Solve $9x^2 + 12x + 4 = 0$.**

  Using $a = 9$, $b = 12$, and $c = 4$, the Quadratic Formula gives:

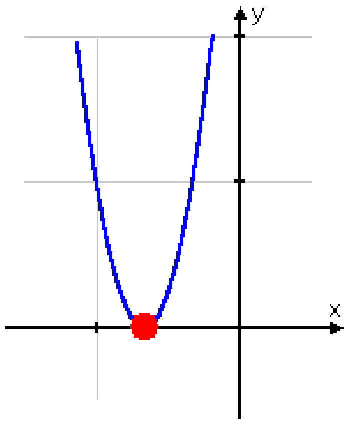
$$x = \frac{-(12) \pm \sqrt{(12)^2 - 4(9)(4)}}{2(9)}$$
$$= \frac{-12 \pm \sqrt{144-144}}{18} = \frac{-12 \pm \sqrt{0}}{18}$$
$$= \frac{-12 \pm 0}{18} = \frac{-12}{18} = \frac{6(-2)}{6(3)} = -\frac{2}{3} = x$$

Then the answer is $x = {}^{-2}/_{3}$

In the previous examples, I had obtained two solutions because of the "plus-minus" part of the Formula. In this case, though, the square root was reduced to zero, so the plus-minus was not important.

This solution is called a "repeated" root.

This is what the graph looks like:



EXAMPLE 4. Solve $3x^2 + 4x + 2 = 0$.

The Quadratic Formula always works; in this case, $a = 3$, $b = 4$, and $c = 2$:

$$x = \frac{-(4) \pm \sqrt{(4)^2 - 4(3)(2)}}{2(3)}$$
$$= \frac{-4 \pm \sqrt{16-24}}{6} = \frac{-4 \pm \sqrt{-8}}{6} = x$$

At this point, I have a negative number inside the square root. If you haven't learned about [complex numbers](#) yet, then you will have to stop here, and the answer will be "no solution".

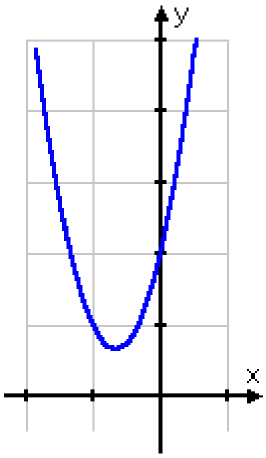
Do you know what a complex number is? If you are curious to know about complex numbers, you can read the following link:

http://www.purplemath.com/modules/complex.htm

but, just to finish our example, we can continue the calculations:

$$\frac{-4 \pm \sqrt{-8}}{6} = \frac{-4 \pm 2\sqrt{-2}}{6} = \frac{-2 \pm \sqrt{2}i}{3} = -\frac{2}{3} \pm \frac{\sqrt{2}}{3}i = x$$

Here's the graph:

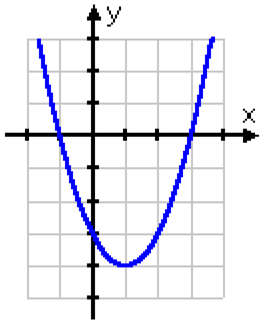


This is always true: if you get a negative value inside the square root, then there will be no *real number* solution, and therefore no *x*-intercepts.

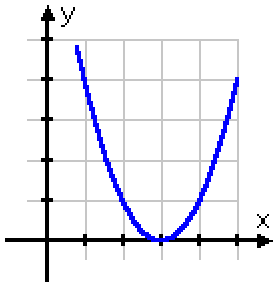
SUMMING UP:

This relationship between the value inside the square root (**the discriminant**), the type of solutions (two different solutions, one repeated solution, or no real solutions), and the number of *x*-intercepts (on the corresponding graph) of the quadratic is summarised in this table:

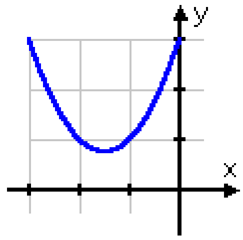
x^2-2x-3	x^2-6x+9	x^2+3x+3
$x=\frac{2\pm\sqrt{(-2)^2-4(-3)}}{2}=\frac{2\pm\sqrt{4+12}}{2}=\frac{2\pm\sqrt{16}}{2}=\frac{2\pm 4}{2}=\frac{-2}{2},\ \frac{6}{2}=-1,\ 3$	$x=\frac{6\pm\sqrt{(-6)^2-4(9)}}{2}=\frac{6\pm\sqrt{36-36}}{2}=\frac{6\pm\sqrt{0}}{2}=\frac{6\pm 0}{2}=3$	$x=\frac{-3\pm\sqrt{(3)^2-4(3)}}{2}=\frac{-3\pm\sqrt{9-12}}{2}=\frac{-3\pm\sqrt{-3}}{2}=-\frac{3}{2}\pm\frac{\sqrt{3}i}{2}$
a positive number inside the square root	zero inside the square root	a negative number inside the square root
two real solutions	one (repeated) real solution	two complex solutions



two distinct x-intercepts



one (repeated) x-intercept



no x-intercepts

Therefore, studying the sign of the discriminant we will know how many solutions a cuadratic equation has got.

The discriminant is $\Delta = b^2 - 4ac$.

4.2. INCOMPLETE CUADRATIC EQUATION.

If the coefficient "b" or "c" is zero, we have got an incomplete cuadratic equation. We can always use the Quadratic Formula and it works, but in these cases, there is an easier method to solve it. For example:

COMPLETE CUADRATIC EQUATION	INCOMPLETE CUADRATIC EQUATIONS
$x^2 - 5x + 6 = 0$	$2x^2 - 7x = 0$ | $5x^2 - 4 = 0$ | $8x^2 = 0$

Continue reading to know the new method in each case.

4.2.1. Solving cuadratic equation when c=0, $ax^2 + bx = 0$.

You have to extract common factor and one solution will be 0.

Example: Solve $3x^2 + 4x = 0$. Extracting common factor:

$$x(3x+4) = 0 \Rightarrow \begin{cases} x = 0 \\ 3x + 4 = 0 \Rightarrow 3x = -4 \Rightarrow x = -4/3 \end{cases}$$

So, the solutions are x= 0 and x = -4/3.

4.2.2. Solving cuadratic equation when b = 0, $ax^2 + c = 0$.

You have to find the value of x^2 and do the square root.

Example: Solve $4x^2 - 1 = 0$.

$$4x^2 = 1 \Rightarrow x^2 = \frac{1}{4} \Rightarrow x = \pm\sqrt{\frac{1}{4}} = \pm\frac{1}{2}$$

So, the solutions are $x_1 = -\frac{1}{2},\ x_2 = \frac{1}{2}$.

Special case: Would you know what the solutions are if the cuadratic equation is $ax^2 = 0$?

Try it on your own.

EXERCISE 1. Solve in your mind:

a) $x^2 = 25$

b) $x^2 = 0$

c) $x^2 = 49$

d) $5x^2 = 0$

e) $x^2 - 1 = 0$

EXERCISE 2. Solve the following equations:

a) $x^2 - 6x = 0$

b) $x^2 - 16 = 0$

c) $x^2 - 5x + 6 = 0$

d) $x^2 + 5x = 0$

e) $x^2 - 25 = 0$

f) $x^2 - 9x = 0$

g) $x^2 - 4x + 4 = 0$

h) $x^2 + 8x = 0$

i) $4x^2 - 81 = 0$

j) $2x^2 - 3x - 20 = 0$

k) $8x^2 - 2x - 3 = 0$

l) $x(x - 3) = 10$

ll) $(x + 2)(x + 3) = 6$

m) $(2x - 3)^2 = 8x$

n) $2x(x - 3) = 3x(x - 1)$

ñ) $\frac{3x}{2} - \frac{x^2 + x}{2} = \frac{3}{8}$

o) $\frac{9x - 4}{10} - x + \frac{x^2 + 2}{30} = 1$

5. PROBLEMS WITH EQUATIONS.

First, think about these questions:

- Calculate the length of the side of a square whose area is 16 m^2.

- Calculate three integers in a row whose addition is 12.

HOW TO SOLVE PROBLEMS WITH EQUATIONS.

Most of the time when someone says "**word problems**" there is automatic panic. But word problems do not have to be the worst part of a math class. By setting up a system and following it, you can solve word problems. So what should you do? Here are some recommended steps:

1. Read the problem carefully and figure out what it is asking you to find.

Usually, but not always, you can find this information at the end of the problem.

2. Assign a variable to the quantity you are trying to find.

Most people choose to use x, but feel free to use any variable you like. For example, if you are being asked to find a number, some students like to use the variable n.

3. Write down what the variable represents.

At the time you decide what the variable will represent, you may think there is no need to write that down in words. However, by the time you read the problem several more times and solve the equation, it is easy to forget where you started.

4. Re-read the problem and write an equation for the quantities given in the problem.

This is where most students have the most trouble. The only way to really master this step is through lots of practice. Be prepared to do a lot of problems.

5. Solve the equation.

The examples done in this lesson will be linear equations.

6. Answer the question in the problem.

Just because you found an answer to your equation does not necessarily mean you are finished with the problem. Many times you will need to take the answer you get from the equation and use it in some other way to answer the question originally given in the problem.

7. Check your solution.

Your answer should not only make sense logically, but it should also make the equation true. If you are asked for a time value and end up with a negative number, this should indicate that you've made an error somewhere. If you are asked how fast a person is running and give an answer of 700 miles per hour, again you should be worried that there is an error. If you substitute these unreasonable answers into the equation you used in step 4 and it makes the equation true, then you should re-think the validity of your equation.

A little summary to know the procedure to solve a problem is:

1ST Understand it: Read the instructions as many times as you need to. Identify unknown information and questions.

2ND Work on it: Write the relation using an equation and solve it.

3RD Solve and check it: Write the answers to the questions and check if they fulfill the given conditions.

Let's practice:

EXAMPLE 1: When 6 is added to four times a number, the result is 50. Find the number.

Step 1: What are we trying to find? A number.
Step 2: Assign a variable for the number. Let's call it n. (you can also use x)
Step 3: Write down what the variable represents.
Let n = a number
Step 4: Write an equation.
We are told 6 is added to 4 times a number. Since n represents the number, four times the number would be $4n$. If 6 is added to that, we get $6+4n$. We know that answer is 50, so now we have an equation $6 + 4n = 50$.
Step 5: Solve the equation.

$$6+4n=50$$

$$4n=44$$

$$n=11$$

Step 6: Answer the question in the problem

The problem asks us to find a number. We decided that n would be the number, so we have n = 11.

The number we are looking for is 11.

Step 7: Check the answer.

The answer makes sense and checks in our equation from Step 4.

$$6+4(11)=6+44=50$$

Here you have a summary table with main ideas to attack the most important kinds of problems:

<table>
<tr><th>Problemas</th><th>Conviene recordar</th></tr>
<tr><td>Numéricos</td><td>Intenta asociar la incógnita con el número menor.
Un número par es 2x. Un número impar es 2x + 1
El 15% de x es 0,15x</td></tr>
<tr><td>Geométricos</td><td>Haz siempre el dibujo.</td></tr>
<tr><td>Edades</td><td>Haz una tabla:
<table>
<tr><th></th><th>Hoy</th><th>Dentro de 10 años</th></tr>
<tr><td>Hijo</td><td>x</td><td>x + 10</td></tr>
<tr><td>Madre</td><td>x + 30</td><td>x + 40</td></tr>
</table></td></tr>
<tr><td>Mezclas</td><td>Haz una tabla:
<table>
<tr><th></th><th>Sustancia A</th><th>Sustancia B</th><th>Mezcla</th></tr>
<tr><td>Precio (€/kg)</td><td>5</td><td>3,5</td><td>4,5</td></tr>
<tr><td>Masa (kg)</td><td>x</td><td>100 – x</td><td>100</td></tr>
<tr><td>Dinero (€)</td><td colspan="3">5x + 3,5(100 – x) = 4,5 · 100</td></tr>
</table></td></tr>
<tr><td>Relojes</td><td>La velocidad del minutero es 12 veces mayor que la velocidad de la aguja horaria, es decir, mientras la aguja horaria da una vuelta, el minutero da 12</td></tr>
<tr><td>Móviles</td><td>Haz siempre un gráfico:
Espacio = velocidad · tiempo; (e = vt)</td></tr>
<tr><td>Ecuaciones de 2º grado</td><td>Comprueba las soluciones.
Rechaza las soluciones de la ecuación, que no lo sean del problema.</td></tr>
</table>

Attention: If you want to ensure you understand everything about quantities and their relationship instead of numbers, visit the following webpage:

http://www.homeschoolmath.net/teaching/teach-solve-word-problems.php

EXERCISE 1: The base of a rectangle is 8 cm long bigger than it is high. If its perimeter is 64 cm., calculate the dimensions of the rectangle.

EXERCISE 2. We mix coffee at € 4.8/kg with coffee at € 7.2/kg. If we want to get 60 kg of blend at € 6.5/kg, how many kilograms of each type of coffee must we mix?

EXERCISE 3. Juan's mother is 26 years older than her son Juan, and in 10 years, his mother will be twice as old as Juan. How old are they?

EXERCISE 4. Calculate two numbers, whose difference is 5, such that the sum of their squared values is 73.

EXERCISE 5. The sum of the squared of two numbers in a row is 181. Calculate these numbers.

EXERCISE 6. Calculate the dimensions of a rectangular plot if it is 3 dam longer than it is wide and its area is 40 dam^2.