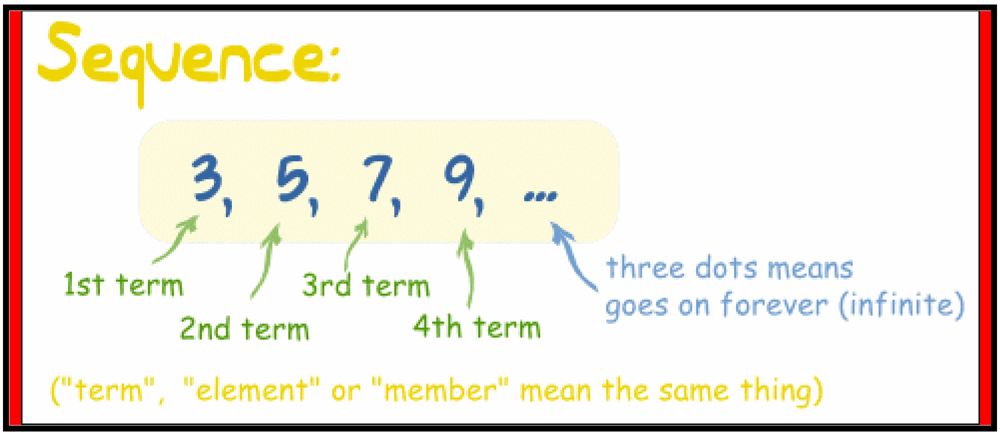
UNIT 3. SEQUENCES

1. WHAT IS A SEQUENCE?

A Sequence is a set of things (usually numbers) that are in order.



Infinite or Finite

If the sequence goes on forever it is called an infinite sequence, otherwise it is a finite sequence

Examples

- {1, 2, 3, 4 ,...} is a very simple sequence (and it is an infinite sequence)
- {20, 25, 30, 35, ...} is also an infinite sequence
- {1, 3, 5, 7} is the sequence of the first 4 odd numbers (and is a finite sequence)
- {4, 3, 2, 1} is 4 to 1 backwards
- {1, 2, 4, 8, 16, 32, ...} is an infinite sequence where every term doubles
- {a, b, c, d, e} is the sequence of the first 5 letters alphabetically
- {f, r, e, d} is the sequence of letters in the name "fred"
- {0, 1, 0, 1, 0, 1, ...} is the sequence of alternating 0s and 1s (yes they are in order, it is an alternating order in this case)

In Order

When we say the terms are "in order", we are free to define **what order that is**! They could go forwards, backwards ... or they could alternate ... or any type of order you want!

Like a Set

A Sequence is like a Set, except:

- the terms are **in order** (with Sets the order does not matter)

- the same value can appear many times (only once in Sets)

Example: {0, 1, 0, 1, 0, 1, ...} is the **sequence** of alternating 0s and 1s.

The **set** would be just {0,1}

Notation;

Sequences also use the same notation as sets: list each element, separated by a comma, and then put curly brackets around the whole thing.	**{3, 5, 7, ..}**

The curly brackets { } are sometimes called "set brackets" or "braces".

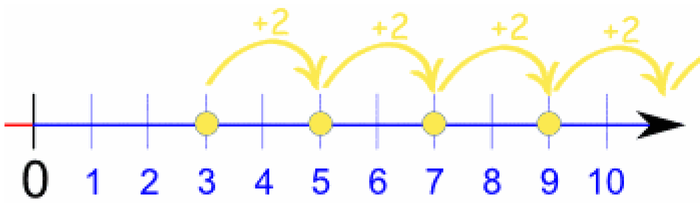
2. WAYS TO GIVE A SEQUENCY.

A Sequency can be given using a rule, a formula or giving first terms, look at the following seccions and examples:

A Rule

A Sequence usually has a **Rule**, which is a way to find the value of each term.

Example: the sequence {3, 5, 7, 9, ...} starts at 3 and jumps 2 every time:



Exercise 1. Calculate the first ten terms in the following sequences:

a) $3, 8, 13, 18\ldots$

b) $8, 4, 0, -4\ldots$

c) $2, -2, 2, -2\ldots$

d) $1/2, 1/4, 1/6, 1/8\ldots$

Exercise 2. Calculate the first ten terms in the following sequences:

a) $2, 1, 2, 4, 2, 7\ldots$

b) $1, 1, 2, 3, 5, 8\ldots$

c) $2, 1, 4, 3, 6, 5\ldots$

d) $1, -2, 4, -8\ldots$

As a Formula

Saying "*starts at 3 and jumps 2 every time*" is fine, but it doesn't help us calculate the:

- 10^{th} term,

- 100th term, or
- ***n***th term, where ***n*** could be any term number we want.

So, we want a formula with "n" in it (where n is any term number).

So, What Would A Rule For {3, 5, 7, 9, ...} Be?

Firstly, we can see the sequence goes up 2 every time, so we can **guess** that a Rule will be something like "2 times n" (where "n" is the term number). Let's test it out:

Test Rule: 2n

n	Term	Test Rule
1	3	2**n** = 2×**1** = **2**
2	5	2**n** = 2×**2** = **4**
3	7	2**n** = 2×**3** = **6**

That **nearly** worked ... but it is **too low** by 1 every time, so let us try changing it to:

Test Rule: 2n+1

n	Term	Test Rule
1	3	2**n**+1 = 2×**1** + 1 = 3
2	5	2**n**+1 = 2×**2** + 1 = 5
3	7	2**n**+1 = 2×**3** + 1 = 7

That Works!

So instead of saying "starts at 3 and jumps 2 every time" we write this:

2n+1

Now we can calculate, for example, the **100th term**:

2 × 100 + 1 = **201**

Exercise 3. Calculate the first four terms in the following sequences:

a) $a_n = 3n + 2$

b) $a_n = (n + 1)^2$

c) $a_n = 3 \cdot 2^n$

d) $a_n = (-2)^n$

Exercise 4. Calculate the first four positive terms in the following sequences and try to get mentally a formula for each one. (In Spanish: formula del término general)

a) Sequence of odd numbers:

b) Sequence of even numbers:

c) Sequence of multiples of number 5:

d) Sequence of perfect squares:

e) Sequence of perfect cubes:

Many Rules

But mathematics is so powerful we can find **more than one Rule** that works for any sequence.

Example: the sequence {3, 5, 7, 9, ...}

We have just shown a Rule for {3, 5, 7, 9, ...} is: **2n+1**

And so we get: {3, 5, 7, 9, 11, 13, ...}

But can we find another rule?

How about **"odd numbers without a 1 in them"**:

And we would get: {3, 5, 7, 9, 23, 25, ...}

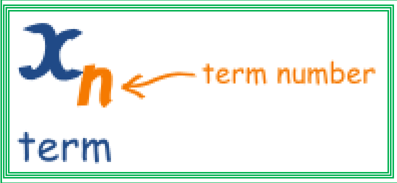
A completely different sequence!

And we could find more rules that match **{3, 5, 7, 9, ...}**. Really we could.

So it is best to say "A Rule" rather than "The Rule" (unless you know it is the right Rule).

Notation

To make it easier to use rules, we often use this special style:



- x_n is the term
- **n** is the term number

Example: to mention the "5th term" you just write: x_5

So a rule for **{3, 5, 7, 9, ...}** can be written as an equation like this:

$$x_n = 2n+1$$

And to calculate the 10th term we can write:

$$x_{10} = 2\mathbf{n}+1 = 2\times\mathbf{10}+1 = 21$$

Can you calculate x_{50} (the 50th term) doing this?

Here is another example:

Solved example: Calculate the first 4 terms of this sequence:

$$\{a_n\} = \{ (-1/n)^n \}$$

Calculations:

- $a_1 = (-1/1)^1 = -1$
- $a_2 = (-1/2)^2 = 1/4$
- $a_3 = (-1/3)^3 = -1/27$
- $a_4 = (-1/4)^4 = 1/256$

Answer:

$$\{a_n\} = \{ -1, 1/4, -1/27, 1/256, \dots \}$$

3. SPECIAL SEQUENCES: ARITHMETIC SEQUENCES AND GEOMETRIC SEQUENCES.

Now let's look at some special sequences, and their rules.

3.1. Arithmetic Sequences

In an Arithmetic Sequence **the difference between one term and the next is a constant**.

In other words, you just add some value each time ... on to infinity.

Example:

1, 4, 7, 10, 13, 16, 19, 22, 25, ...

This sequence has a difference of 3 between each number.
Its Rule is $x_n = 3n-2$

In General you could write an arithmetic sequence like this:

$$\{a, a+d, a+2d, a+3d, \dots \}$$

where:

- **a** is the first term, and
- **d** is the difference between the terms (called the **"common difference"**)

And you can make the rule by:

GENERAL TERM OF ARITHMETIC SEQUENCE

$$x_n = a + d(n-1)$$

(We use "n-1" because **d** is not used in the 1st term).

Exercise 5. Look at the following sequences:

a) 5, 9, 13, 17… b) 6, 3, 0, −3…

c) 2/3, 1/3, 0, − 1/3… d) 1/2, 1, 3/2, 2…

a) Are they arithmetic sequences? If they are, find the common difference in each case.

b) Find the general formula (término general) in each case.

Exercise 6. Write the general formula and the first three terms in an arithmetic sequence knowing the first term is $a_1 = 6$ and d = 2.5:

Exercise 7. In the sequence 5, 9, 13, 17,… which term is the number 49?

Exercise 8. In the arithmetic sequence we know that $a_5 = 19$, $a_8 = 28$. Calculate the common difference and the first term.

➢ ADDITION OF TERMS IN AN ARITHMETIC SEQUENCE.

The addition of the first "n" terms of an arithmetic sequence is represented by S_n and we can calculate it using the following formula:

$$\boxed{S_n = \frac{a_1 + a_n}{2} \cdot n}$$

Solved example: Calculate the addition of the first 25 terms of the arithmetic sequence which is given by $a_n = 7n - 5$.

Using general formula,

$$S_n = \frac{a_1 + a_n}{2} \cdot n \Rightarrow S_{25} = \frac{a_1 + a_{25}}{2} \cdot 25$$

So, we have to calculate a_1 and a_{25}:

$$a_n = 7n - 5$$
$$a_1 = 7 \cdot 1 - 5 = 2$$
$$a_{25} = 7 \cdot 25 - 5 = 170$$

Therefore

$$S_{25} = \frac{2 + 170}{2} \cdot 25 = 2\,150$$

Exercise 8. Calculate the addition of the first 25 terms of the arithmetic sequence which general term is $a_n = 2n + 6$.

Exercise 9. Calculate the addition of the first 25 terms of the arithmetic sequence which general term is $a_n = 3n/2 + 2$.

3.2. Geometric Sequences

In a Geometric Sequence each term is found by **multiplying** the previous term by a **constant**.

Example:

2, 4, 8, 16, 32, 64, 128, 256, ...

This sequence has a factor of 2 between each number. Its Rule is $x_n = 2^n$

In General you could write an arithmetic sequence like this:

$$\{ a, a \cdot r, a \cdot r^2, a \cdot r^3, \ldots \}$$

where:

- **a** is the first term, and
- **r** is the factor between the terms (called the **"common ratio"**)

Note: **r** should not be 0 or 1.

- When **r=0**, you get the sequence {a,0,0,...} which is not geometric
- When **r=1**, you get the sequence {a,a,a,...} which is not geometric

And the rule is:

GENERAL TERM OF GEOMETRIC SEQUENCE

$$x_n = ar^{(n-1)}$$

(We use "n-1" because ar^0 is the 1st term)

➢ ADDITION OF TERMS IN A GEOMETRIC SEQUENCE.

The addition of the first "n" terms of a geometric sequence is represented by S_n and we can calculate it using the following formula:

$$S_n = \frac{a_n \cdot r - a_1}{r - 1}, \; r \neq 1$$

Solved example: Calculate the addition of the first 7 terms of the geometric sequence which is given by ***6, 12, 24, 48,…***

$$S_n = \frac{a_n \cdot r - a_1}{r - 1} \Rightarrow S_7 = \frac{a_7 \cdot r - a_1}{r - 1}$$

$$a_1 = 6, \; r = \frac{12}{6} = 2$$

$$a_n = 6 \cdot 2^{n-1} \Rightarrow a_7 = 6 \cdot 2^{7-1} = 6 \cdot 2^6 = 6 \cdot 64 = 384$$

$$S_7 = \frac{384 \cdot 2 - 6}{2 - 1} = 762$$

$\boxed{384}\,\boxed{\times}\,\boxed{2}\,\boxed{-}\,\boxed{6}\,\boxed{=}\,\boxed{762}$

➢ ADDITION OF TERMS IN A DECRESING GEOMETRIC SEQUENCE.

The addition of the first all terms of a geometric sequence with $|r| < 1$ is equal to:

$$\boxed{S = \frac{a_1}{1 - r}}$$

Solved example: Calculate the addition of all terms of the geometric sequence which is given by 3, 1, 1/3, …

$$S = \frac{a_1}{1 - r};\ a_1 = 3,\ r = \frac{1}{3}$$

$$S = \frac{3}{1 - 1/3} = \frac{3}{2/3} = 3 : \frac{2}{3} = 3 \cdot \frac{3}{2} = \frac{9}{2}$$

Exercise 10. Calculate the general term of the following geometric sequences:

a) 5, 15, 45, 135…

b) 6, 3, 3/2, 3/4…

Exercise 11. In a geometric sequence the first term, $a_1 = 4$ and the common ratio (in Spanish: razón) r = 5. Calculate:

a) a_6
b) a_{10}
c) a_n

Exercise 12. In the geometric sequence 2, 4, 8, 16, 32, … Which term is equal to 1024?

Exercise 13. Find out the common ratio (razón) of a geometric sequence knowing that $a_4 = 135$; and $a_6 = 1215$.

Exercise 14. Find out the addition of first 10 terms of the following geometric sequences:

a) 2, 14, 98, 686…

b) 3, −6, 12, −24…

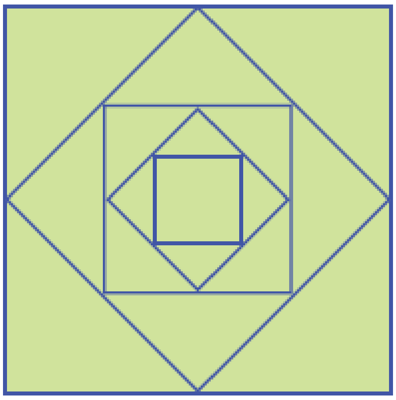
Exercise 15. Find out the addition of all terms of the following geometric sequences:

a) 1/5, 1/25, 1/125, 1/625…

b) 3, 2, 4/3, 8/9, 16/27…

Exercise 16. The addition of all terms in a geometric sequence is equal to 6 and its first term is 4. Calculate the common ratio.

Exercise 17. Si en un cuadrado de área 8 m2 se unen los puntos medios, se obtiene otro cuadrado, y así sucesivamente. Calcula la sucesión de las áreas de dichos cuadrados. ¿De qué progresión se trata?



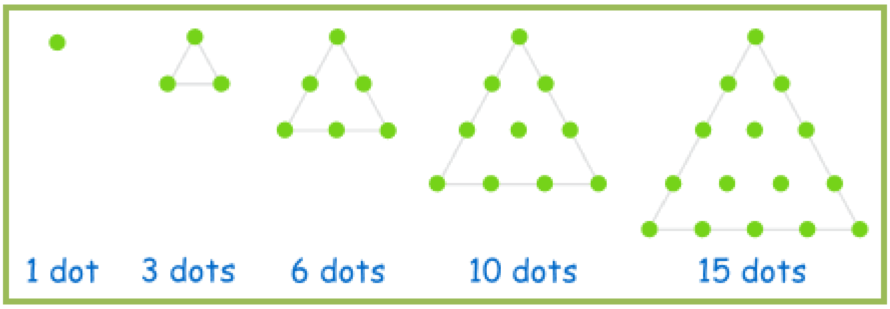
4. OTHER EXAMPLES OF SEQUENCES:

- **Triangular Numbers**

1, 3, 6, 10, 15, 21, 28, 36, 45, ...

This sequence is generated from a pattern of dots which form a triangle.

By adding another row of dots and counting all the dots we can find the next number of the sequence:



But it is easier to use this Rule:

$$x_n = n(n+1)/2$$

Example:

- the 5th Triangular Number is $x_5 = 5(5+1)/2 = \mathbf{15}$,
- and the sixth is $x_6 = 6(6+1)/2 = \mathbf{21}$

Square Numbers

1, 4, 9, 16, 25, 36, 49, 64, 81, ...

The next number is made by squaring where it is in the pattern.

Rule is $x_n = n^2$

Cube Numbers

1, 8, 27, 64, 125, 216, 343, 512, 729, ...

The next number is made by cubing where it is in the pattern.

Rule is $x_n = n^3$

Fibonacci Sequence

This is the Fibonacci Sequence

0, 1, 1, 2, 3, 5, 8, 13, 21, 34, ...

The next number is found by **adding the two numbers before it** together:

- The 2 is found by adding the two numbers before it (1+1)
- The 21 is found by adding the two numbers before it (8+13)
- etc...

Rule is $x_n = x_{n-1} + x_{n-2}$

That rule is interesting because it depends on the values of the previous two terms.

Rules like that are called **recursive** formulas.

The Fibonacci Sequence is numbered **from 0 onwards** like this:

n =	*0*	*1*	*2*	*3*	*4*	*5*	*6*	*7*	*8*	*9*	*10*	*11*	*12*	*13*	*14*	*...*
x_n =	0	1	1	2	3	5	8	13	21	34	55	89	144	233	377	...

Example: term "6" would be calculated like this:

$$x_6 = x_{6-1} + x_{6-2} = x_5 + x_4 = 5 + 3 = 8$$

5. APLICATIONS: SIMPLE INTEREST AND COMPOSITE INTEREST.

5.1. Simple interest.

When you deposit money in a savings account, the bank pays you interest at a certain rate called interest rate. There are two types of interest: simple interest and compound interest. In this lesson we will talk about simple interest. We will use the following important formula:

$$\boxed{I = prt}$$

I - represents the interest (or the amount of money that the bank will pay you for allowing it to use your savings account).

p – represents the principal (the money you initially deposit)

r – represents the interest rate

t – represents the time in years.

Solved Example:

Mario deposits $3,000 in a savings account at the interest rate of 5% per year. How much simple interest will Mario earn in three years?

Solution:

Let's identify the important elements in this problem. The initial deposit of $3000 is the principal (p = $3000). The interest rate, r = 5%. The time is t = 3 years. We have to find the simple interest, I.

In the formula I = prt, replace p, r, and t with the given values.

I = 3000 x 5% x 3

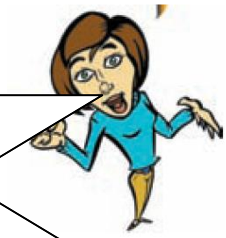
Change 5% to a decimal by moving the decimal point twice to the left:

5% = 0.05

Therefore, I = 3000 x 0.05 x 3 = **$450**.

You can visit this website to wacth an interesting video:

http://www.math-videos-online.com/simple-interest.html



5.2. Compound interest.

- **Important Facts**

Unlike the simple interest, the compound interest pays interest on both the principal and the interest already earned. To find the final balance after a certain number of years, use the following important formula:

$$B = p\,(1 + r)^t$$

B is the final balance
p is the initial principal
r is the interest rate per year
t is the time in years

Solved example:

If you deposit \$5000, in an account that earns 8% compounded annually, find the compound interest at the end of the second year and find the final balance.

Solution:

At the end of the first year, the interest will be

Interest year 1 = 5000 x 8% x 1 = 5000 x 0.08 x 1 = \$400

Add \$400 to the initial principal (\$5000) to find the new principal for the second year: \$5400

The interest at the end of the second year will be:

Interest year 2 = 5400 x 8% x 1 = 5400 x 0.08 x 1 = \$432

Therefore, the total compound interest at the end of the year two will be

$400 + $432 = $832

To find the final balance you can use the formula B = p (1 + r)^t :

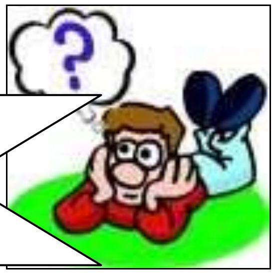
B = p (1 + r)^t = 5000 (1 + 0.08)^2 = 5000 x (1.08)^2 = 5000 x 1.1664 = $5832

If you do not want to use the above formula, you can also find the final balance by adding the compound interest earned at the end of the year two ($432), to the second year principal ($5400).

$432 + $5400 = $5832

You can visit this website to watch an interesting video:

http://www.math-videos-online.com/compound-interest.html



EXERCISES

UNIT 3. SEQUENCES.

Exercise 1. Copy the instructions and solve the following questions visiting the Internet:

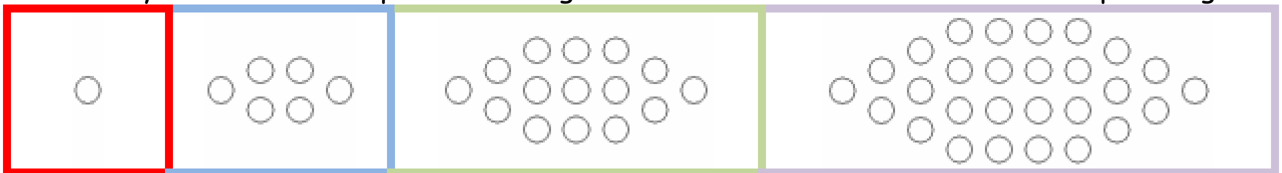
QUESTION 1.
http://www.mathopolis.com/questions/q.php?id=595&site=1&ref=/algebra/sequences-series.html

QUESTION 2.
http://www.mathopolis.com/questions/q.php?id=596&site=1&ref=/algebra/sequences-series.html

QUESTION 3.
http://www.mathopolis.com/questions/q.php?id=597&site=1&ref=/algebra/sequences-series.html

Exercise 2. <u>Make your own Number Patterns</u>

You can make your own number patterns using coins or matchsticks. Here is an example using coins:



Size=1	**Size=2**	**Size=3**	**Size=4**
1 Coin	**6 Coins**	**15 Coins**	**28 Coins**

How many coins would you need when Size=5 ?

Can you make a formula that will tell you how many coins are needed for any Size? For example Size=20? The formula may look something like

$$\text{Coins} = \text{Size} \times \text{Size} + \ldots$$

Exercise 3. PROBLEMS OF SIMPLE INTEREST AND COMPOSITE INTEREST

1. John deposits $7,500 in a savings account, at an interest rate of 6%, for 2 years. How much simple interest will he earn? If the government gets 18% of the interest earned, how much will he earn?

2. Mary deposited €5,000 at an interest rate of 3.5% and she earned €700 interest. How long did she keep the deposit?

3. I deposited €18,000 for 5 years and got €1,500 simple interest. What was the interest rate?

4. €6,500 were deposited at 5% compound interest for 4 years. 18% was paid for taxes. Calculate the final capital if the interests are paid annually.

5. A deposit earning 5% compound interest annually for 4 years generated a final capital of €15,558.48. How much was the initial capital?

ATTENTION: **To revise for your test, you can visit this useful webpage (NUMBER SEQUENCES section): http://www.ixl.com/math/grade/eighth/**

MATHS IS FUN!